Symmetries in Nuclear and Particle Physics

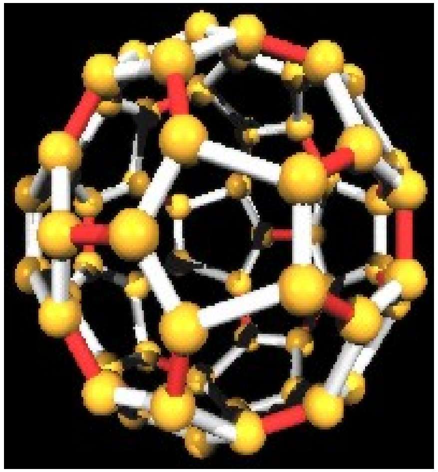
- 1. Symmetries in Physics
- 2. Interacting Boson Model
- 3. Nuclear Supersymmetry
- 4. Quark Model
- 5. Unquenched Quark Model

Roelof Bijker (ICN-UNAM)
bijker@nucleares.unam.mx

1. Symmetries in Physics

- Introduction
- Group theory
  - Lie groups and Lie algebras
  - Conservation laws
- Hydrogen atom
- Dynamical symmetry
- Isospin symmetry in nuclei

Introduction



- Geometric symmetries

  Buckyball with icosahedral symmetry
- Permutation symmetries

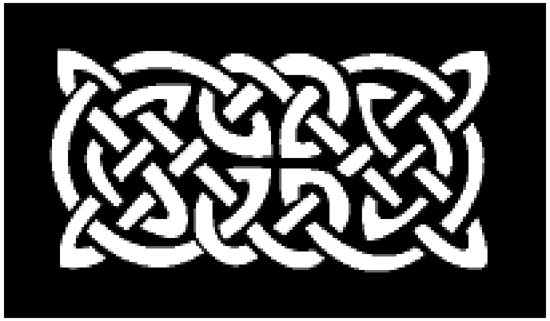
  Fermi-Dirac and Bose-Einstein statistics
- Space-time symmetries

  Rotational invariance in nonrelativistic QM, Lorentz invariance in relativistic QM
- Gauge symmetries

  Dirac equation with external electromagnetic field

Dynamical Symmetries

- Hydrogen atom (Pauli, 1926)
- Isospin symmetry (Heisenberg, 1932)
- Spin-isospin symmetry (Wigner, 1937)
- Pairing, seniority (Racah, 1943)
- Elliott model (Elliott, 1958)
- Flavor symmetry (Gell-Mann, Ne'eman, 1962)
- Interacting boson model (Arima, Iachello, 1974)
- Nuclear supersymmetry (Iachello, 1980)

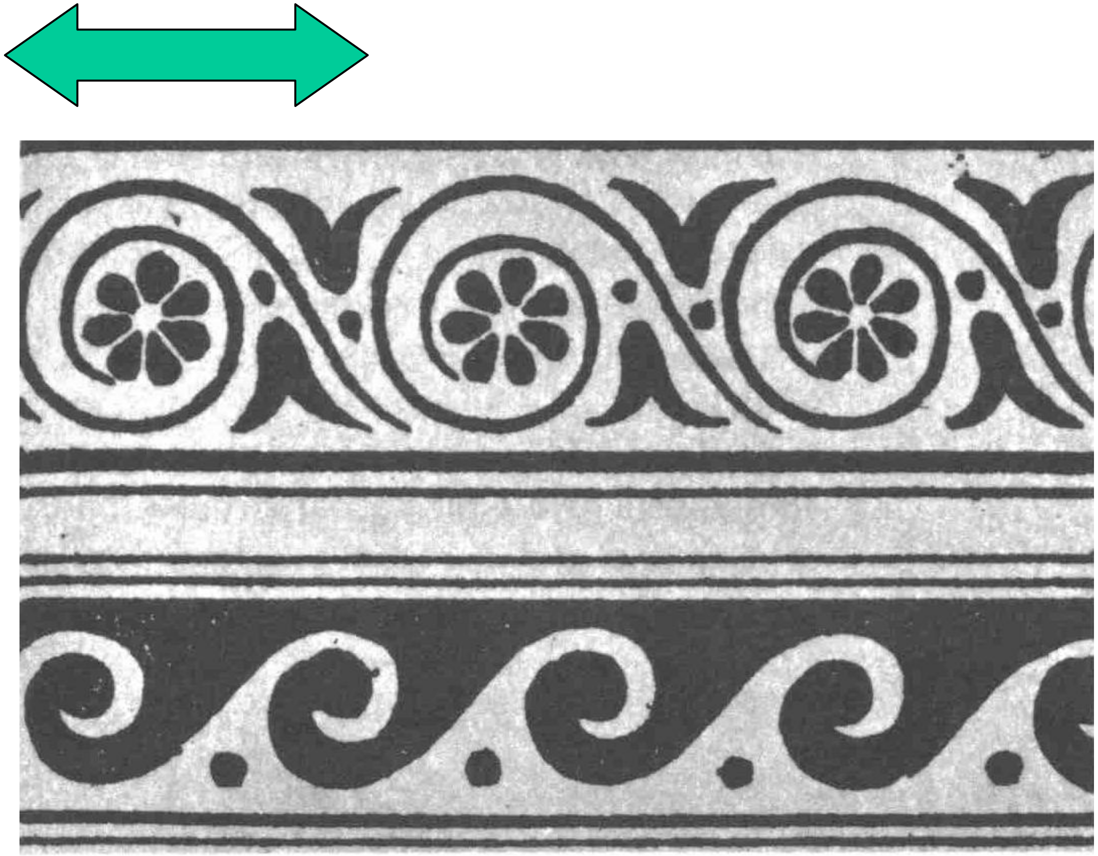


Reflection symmetry

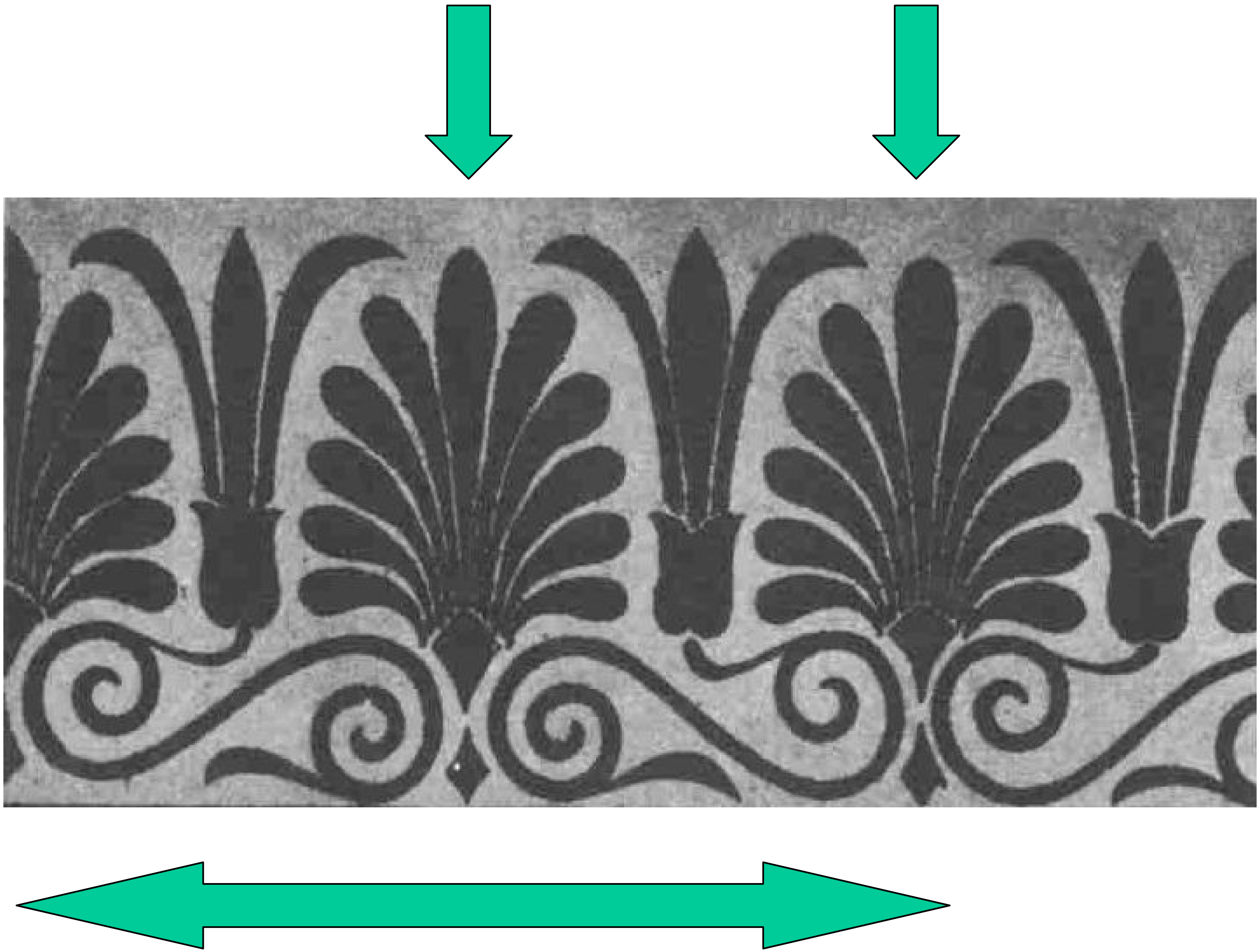
 

Taj Mahal, before and after reflection
about the symmetry axis

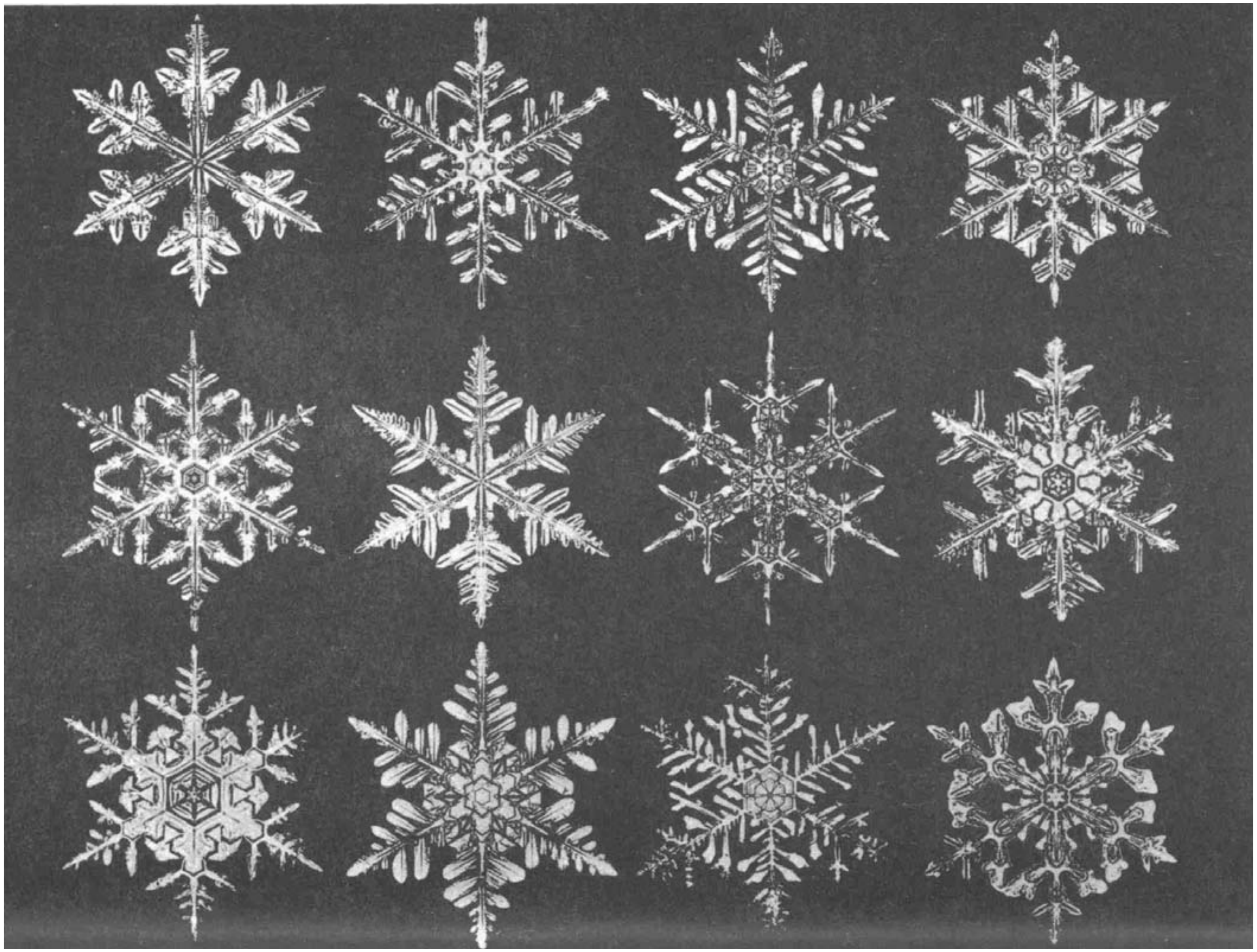
Translation Symmetry

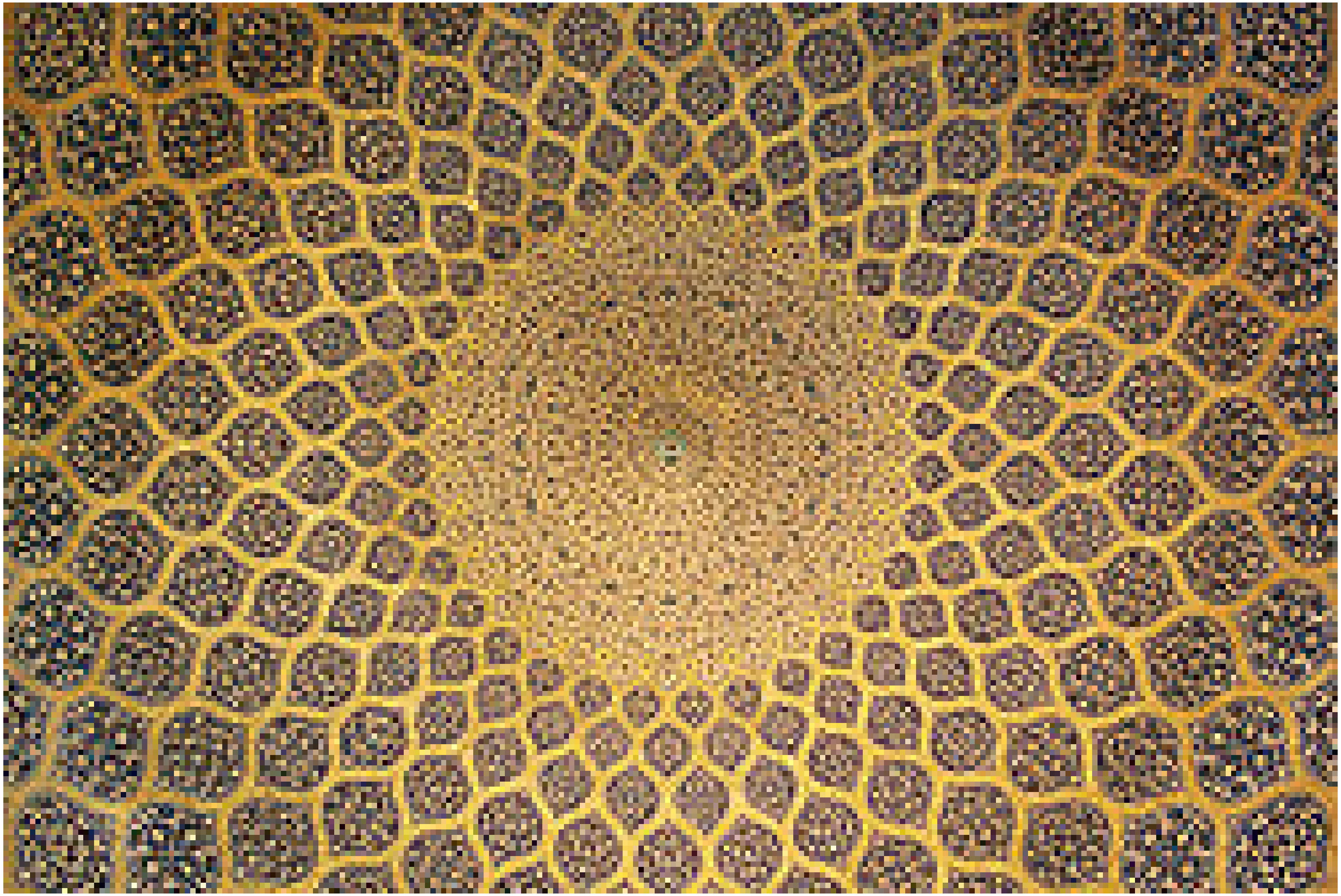


Reflection and Translation Symmetry



Rotation Symmetry





Symmetries in Physics

- Greek origin: « with proportion, with order ».
- Oxford Dictionary of Current English:
  - Symmetry: « ...right correspondence of parts; quality of harmony or balance (in size, design etc.) between parts ».
- Symmetry in physics via group theory (mathematical theory of symmetry)
- Main protagonists
  - Evariste Galois (1831): Galois theory
  - Sophus Lie (1873): Lie algebra
  - Emmy Noether (1918): Noether's theorem

Group Theory

A set of elements $a, b, c, \ldots \epsilon\, G$

and a multiplication operation $\circ$

Closure	$a \circ b \in G$
Associativity	$(a \circ b) \circ c = a \circ (b \circ c)$
Identity	$a \circ e = e \circ a = a$
Inverse	$a \circ a^{-1} = a^{-1} \circ a = e$

Evariste Galois (1811-1832)

Bell - Men of Mathematics
Ch 20: Genius and stupidity

Abelian Groups

Closure	$a \circ b \in G$
Associativity	$(a \circ b) \circ c = a \circ (b \circ c)$
Identity	$a \circ e = e \circ a = a$
Inverse	$a \circ a^{-1} = a^{-1} \circ a = e$
Commutativity	$a \circ b = b \circ a$

Niels Henrik Abel (1802-1829)

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Lie Groups and Lie Algebras

- A Lie group contains an infinite number of elements that depend on a set of continous variables

$$\vec{z}\,' = A(\alpha)\vec{z}\ , \qquad A(\alpha) = A(\alpha_1, \ldots, \alpha_r)$$

- The corresponding Lie algebra is obtained from infinitesimal generators

$$X_k = i\,\frac{\partial A}{\partial \alpha_k}\bigg|_{\alpha=0}$$

- Commutation relations

$$[X_k, X_l] \equiv X_k X_l - X_l X_k = \sum_m c_{kl}^m X_m$$

- Jacobi identity

$$[X_k, [X_l, X_m]] + [X_l, [X_m, X_k]] + [X_m, [X_k, X_l]] = 0$$

Orthogonal Rotations

Example: rotations in space are generated by the angular momentum operator

$$L_j = (\vec{r} \times \vec{p})_j = \epsilon_{jkl} r_k p_l = -i\epsilon_{jkl} r_k \frac{\partial}{\partial r_l}$$

which close under the Lie algebra of SO(3)

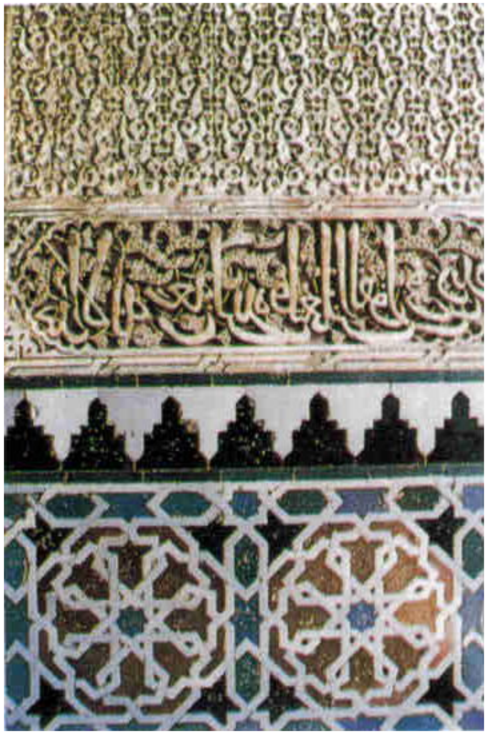
$$[L_j, L_k] = i\epsilon_{jkl} L_l$$

Casimir invariant

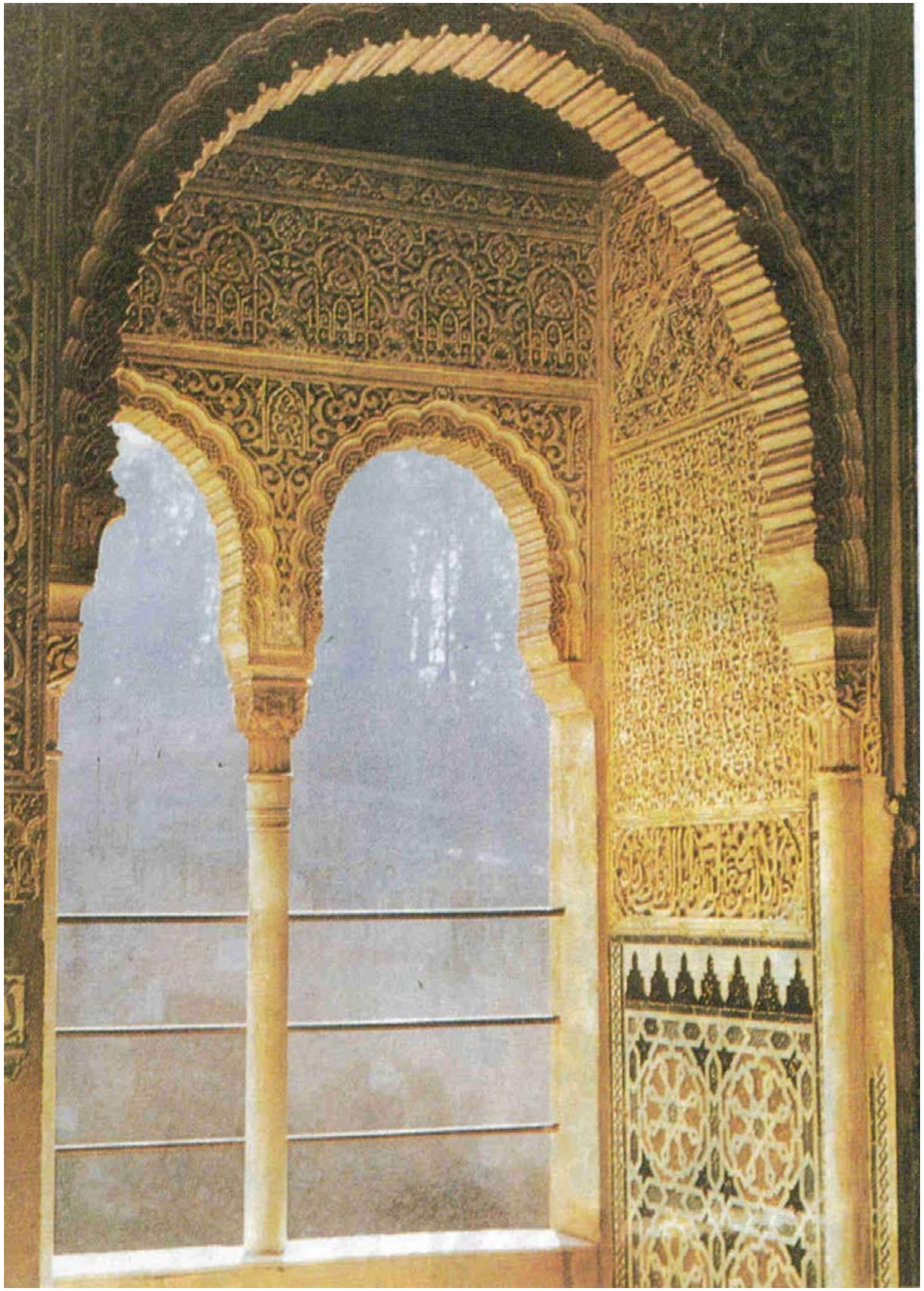
$$\left[\vec{L}^2, L_j\right] = 0$$

Sophus Lie (1842-1899)

Hendrik Casimir (1909-2000)



Alhambra, Spain



Symmetries and Conservation Laws

Emmy Noether
(1882-1935)

Noether's theorem

For every continuous symmetry of the laws of physics, there exists a conservation law

Symmetry	Conservation Law
Translation in space	Linear momentum
Translation in time	Energy
Rotation	Angular momentum

Invariance

If the hamiltonian H is invariant under a Lie group of continuous transformations G, it commutes with the generators of the Lie algebra

$$[H, X_k] = 0 \qquad \forall\, X_k$$

Rotational invariance $\quad [H, L_j] = 0$

$$[H, \vec{L}^2] = 0 \ , \qquad [\vec{L}^2, L_j] = 0$$

Eigenvalue equations

$$\begin{aligned} H\,\psi_{Elm} &= E\,\psi_{Elm} \\ \vec{L}^2\,\psi_{Elm} &= l(l+1)\,\psi_{Elm} \\ L_z\,\psi_{Elm} &= m\,\psi_{Elm} \end{aligned}$$

Conservation Laws

Space-time symmetries

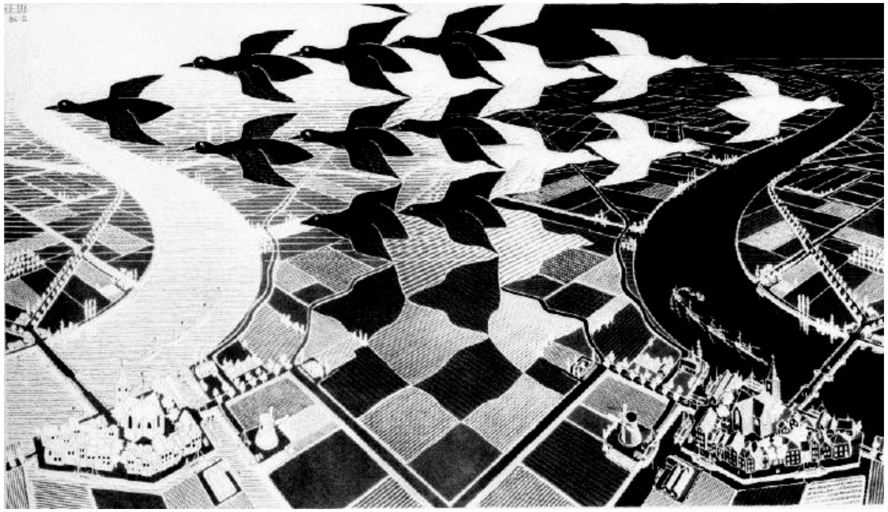
- Energy, momentum and angular momentum

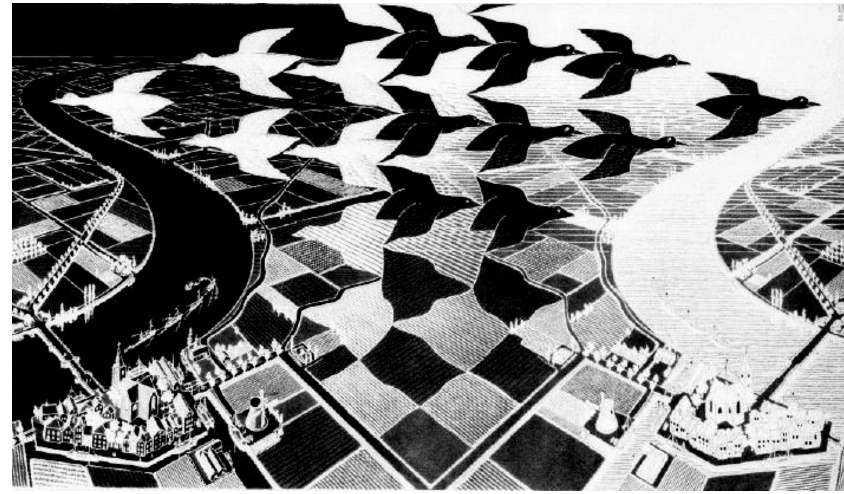
Internal symmetries

- Electric charge, baryon and lepton number
- Isospin and strangeness

Discrete symmetries

- Parity, charge conjugation and time reversal





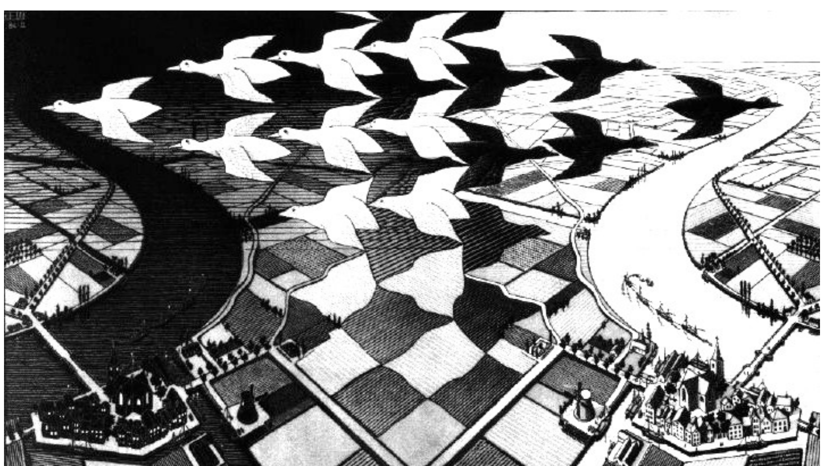
Charge Inversion
Particle-antiparticle
mirror

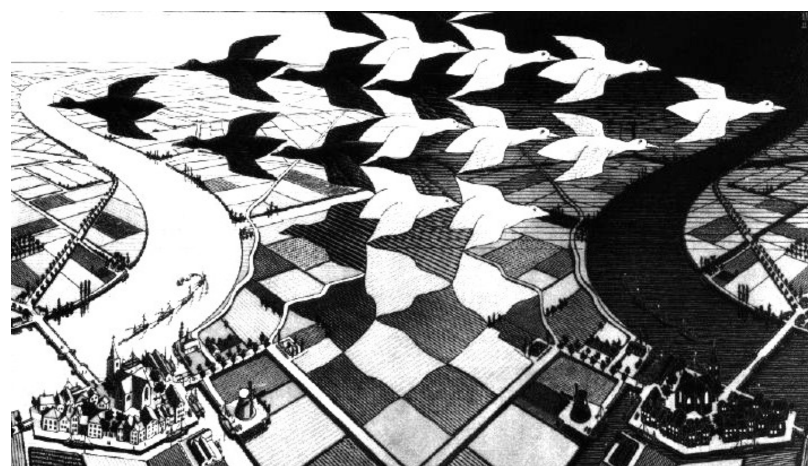
C

P

Parity
Inversion
Spatial
mirror

CP





Consequences of Symmetry

- Conservation laws
- Selection rules
- State labeling $H\,|\Gamma\gamma\rangle = E(\Gamma)\,|\Gamma\gamma\rangle$
- Degeneracy

$$\begin{aligned} H\,|\Gamma\gamma\rangle &= E\,|\Gamma\gamma\rangle \\ &\Rightarrow HX\,|\Gamma\gamma\rangle = EX\,|\Gamma\gamma\rangle \end{aligned}$$

- Action of transformations

$$X\,|\Gamma\gamma\rangle = \sum_{\gamma'} a^{\Gamma}_{\gamma\gamma'}(X)\,|\Gamma\gamma'\rangle$$

The Hydrogen Atom

Hamiltonian $$H = \frac{p^2}{2m} - \frac{e^2}{r}$$

Eigenstates $$\psi_{nlm}(r, \theta, \phi)$$

$$n = 1, 2, \ldots,$$
$$l = 0, 1, \ldots, n - 1$$
$$-l \leq m \leq l$$

Energies $$E_{nlm} = -\frac{me^4}{2n^2}$$

Degeneracy $$\sum_{l=0}^{n-1} (2l + 1) = n^2$$

SO(3) Symmetry

- The hamiltonian of the H-atom commutes with angular momentum operator:

$$[H,\vec{L}] = [H,\vec{r}\times\vec{p}] = 0$$

- The *L* operators generate an SO(3) algebra:

$$[L_j, L_k] = i\epsilon_{jkl}L_l \qquad j,k,l = x,y,z$$

- SO(3) symmetry $\Rightarrow$ degeneracy in m $2l+1$
- What is the origin of the additional degeneracy in l?

Kepler Problem

In analogy with the Kepler problem in classical mechanics introduce quantum mechanical analogue of the Runge-Lenz vector

$$\vec{A} = \frac{1}{2m}(\vec{L} \times \vec{p} - \vec{p} \times \vec{L}) - \frac{e^2}{r}\vec{r}$$

satisfying

$$[H, \vec{A}] = [H, \vec{L}] = 0$$
$$\vec{L} \cdot \vec{A} = \vec{A} \cdot \vec{L} = 0$$
$$\vec{A}^2 = \frac{2H}{m}(\vec{L}^2 + 1) + e^4$$

Pauli (1926)

SO(4) Symmetry

For bound states (E<0) introduce $\vec{B} = \sqrt{-m/2E}\,\vec{A}$

The angular momentum operator and the modified Runge-Lenz vector close under commutation of SO(4)

$$[L_j, L_k] = i\epsilon_{jkl} L_l$$
$$[L_j, B_k] = i\epsilon_{jkl} B_l$$
$$[B_j, B_k] = i\epsilon_{jkl} L_l$$

Introduce $\vec{I} = \frac{1}{2}(\vec{L} + \vec{B})\ , \quad \vec{K} = \frac{1}{2}(\vec{L} - \vec{B})$

satisfying $[I_j, I_k] = i\epsilon_{jkl} I_l\ , \quad [K_j, K_k] = i\epsilon_{jkl} K_l$

$$[I_j, K_k] = 0$$

Then

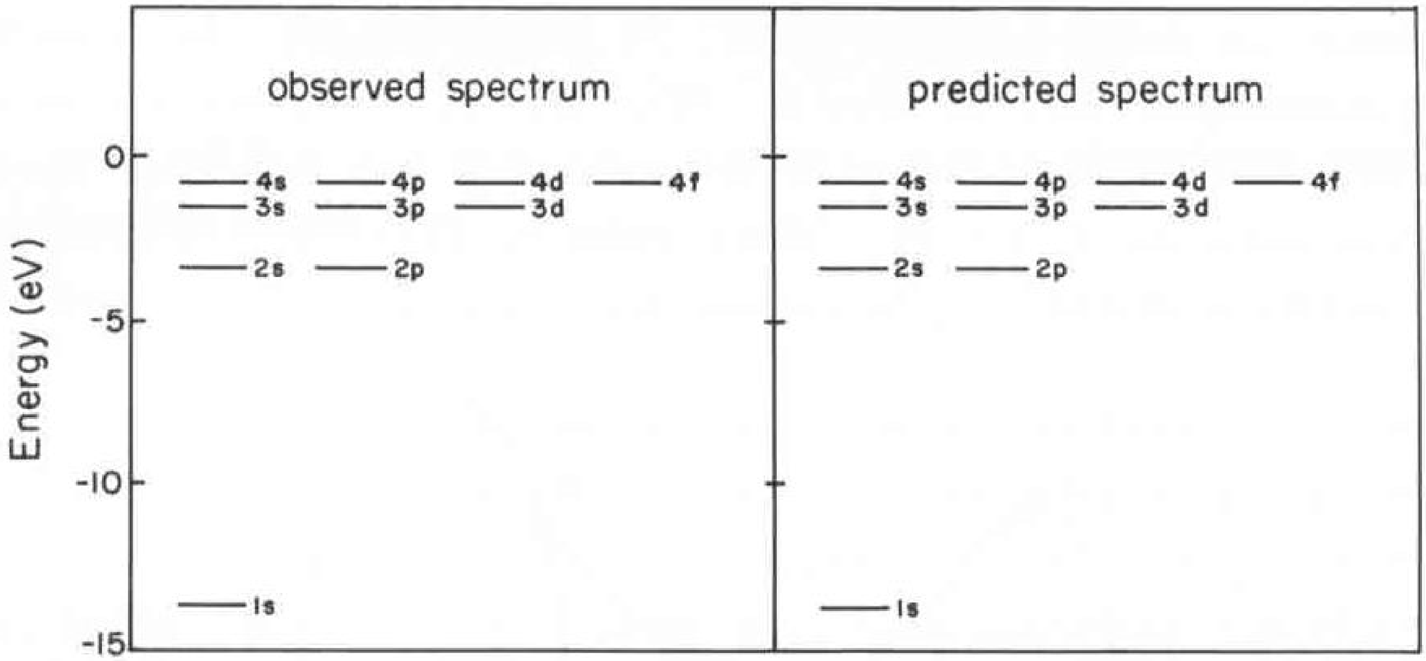
$$\vec{L}\cdot\vec{B} = \vec{I}^2 - \vec{K}^2 = 0$$
$$\frac{1}{2}(\vec{L}^2 + \vec{B}^2) = \vec{I}^2 + \vec{K}^2 = 2\vec{K}^2$$

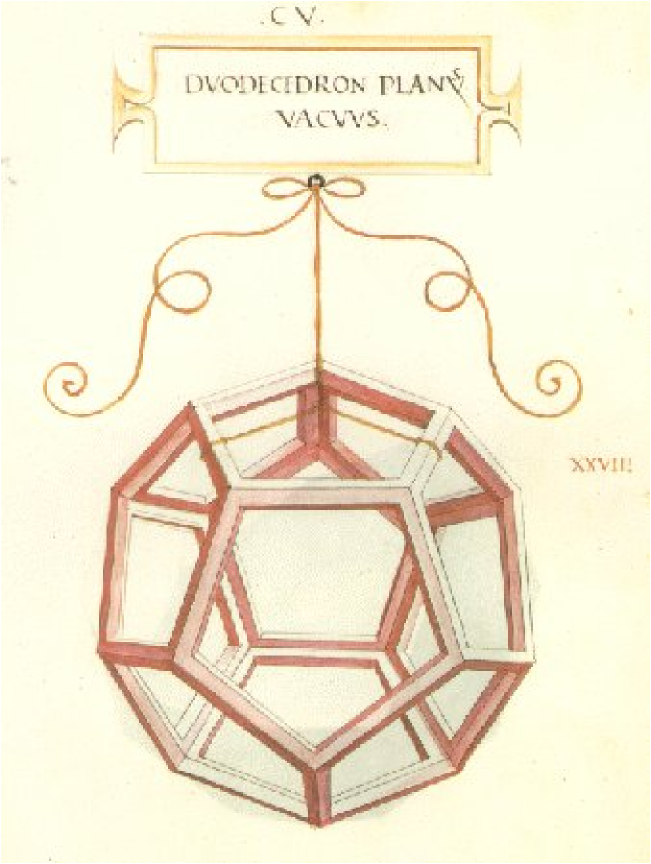
$$-\frac{1}{2} - \frac{me^4}{4E} = 2k(k+1)$$
$$E = -\frac{me^4}{2n^2} \qquad \text{with } n = 2k+1$$

Angular momentum $\vec{L} = \vec{K} + \vec{I}$

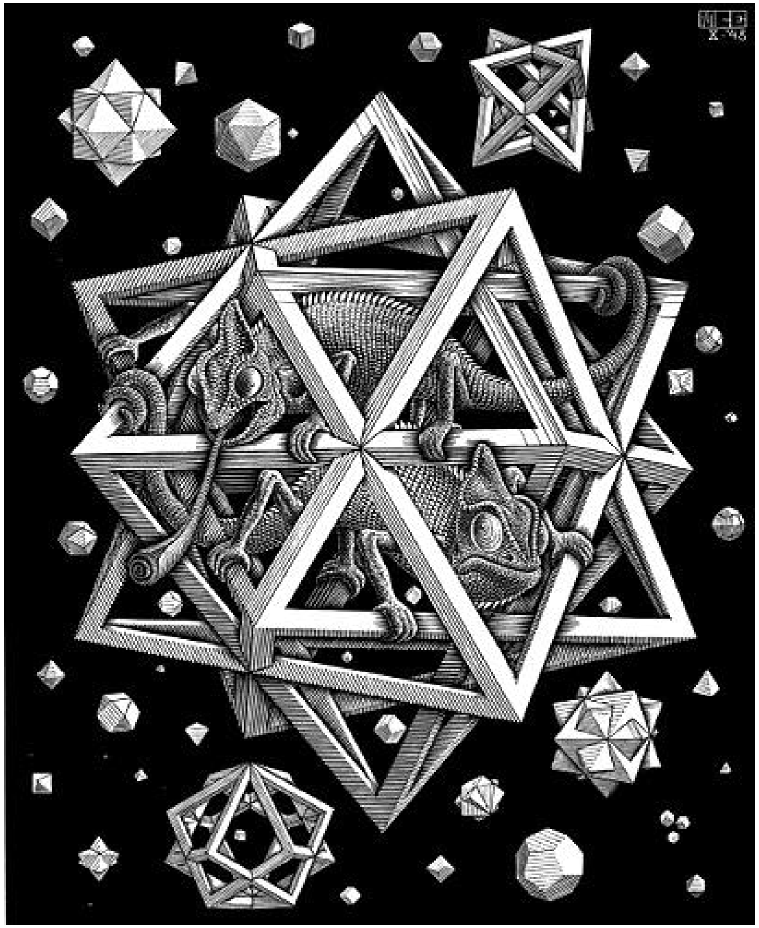
$$0 \le l \le 2k = n-1$$

Energy Spectrum of H-Atom





Da Divina Proportione (1509)
Luca Pacioli (1445-1517)
Leonardo da Vinci (1452-1519)



Maurits Escher (1898-1972)

Dynamical Symmetry

- Chain

Algebras :	G_1	$\supset$	G_2
Generators :	X_1		X_2
Labels :	$\vert \Gamma_1$	,	$\Gamma_2, \gamma\rangle$

- Assume G_1 is a symmetry of H_1

$$[H_1, X_1] = 0 \;, \qquad \forall X_1 \in G_1$$
$$H_1 = \alpha_1 \hat{C}_{G_1}$$
$$E_1 = \alpha_1 C_1(\Gamma_1)$$

- Assume G_2 is a symmetry of H_2

$$[H_2, X_2] = 0 \;, \qquad \forall X_2 \in G_2 \subset G_1$$
$$H_2 = \alpha_2 \hat{C}_{G_2}$$
$$E_2 = \alpha_2 C_2(\Gamma_2)$$

Dynamical Symmetry

- $H=H_1+H_2$ has symmetry G_2 and dynamical symmetry G_1

$$\forall X_2 \in G_2 \qquad [H, X_2] = 0$$
$$\exists X_1 \in G_1 \qquad [H, X_1] \neq 0$$

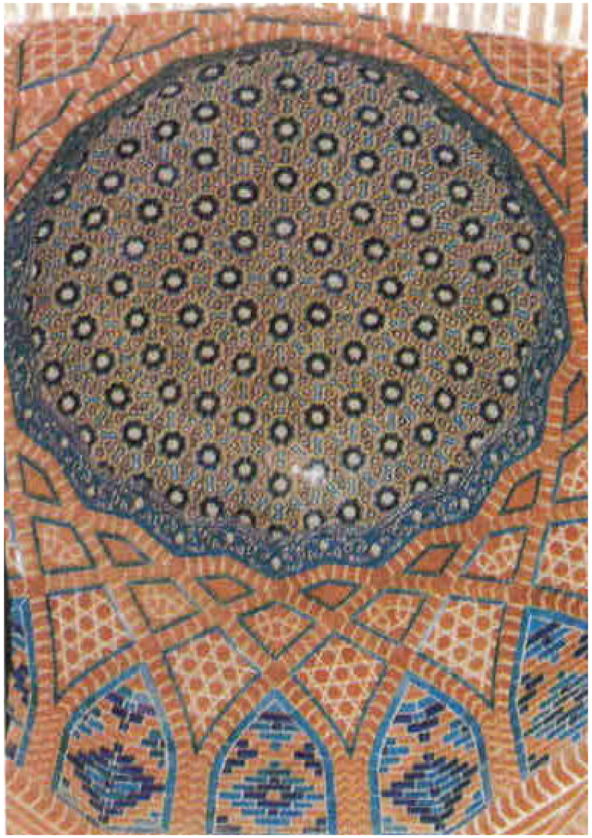
- The eigenstates of H are the same as those of H_1 and are independent of the parameters α_1 and α_2

$$H \left| \Gamma_1 \Gamma_2 \gamma \right\rangle = E \left| \Gamma_1 \Gamma_2 \gamma \right\rangle$$

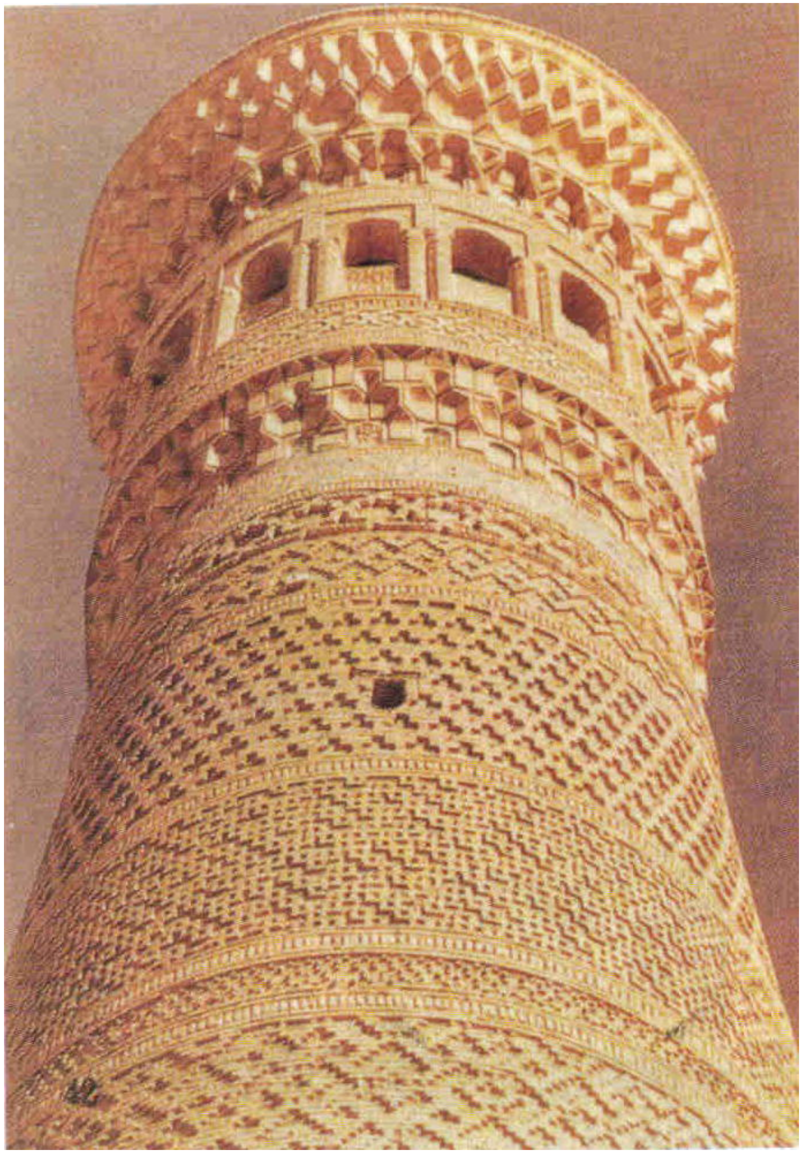
- H splits but does not admix eigenstates of H_1 with eigenvalues

$$E = E_1 + E_2 = \alpha_1 C_1(\Gamma_1) + \alpha_2 C_2(\Gamma_2)$$

- A Dynamical Symmetry arises whenever the Hamiltonian is written in terms of the Casimir operators a chain of subalgebras



Shahjahan Mosque, Pakistan



Kalyan Minaret, Uzbekistan

Isospin Symmetry in Nuclei

- Empirical observations:
  - Almost equal masses of n and p
  - n and p have spin ½
  - Equal (to ~1%) nn, np, pp strong forces.
- This suggests an isospin symmetry of the nuclear hamiltonian

Werner Heisenberg (1932)

Isospin of the Nucleon

Nucleon is an isospin doublet with isospin I=½

$$p: \quad |I, I_3\rangle = \left|\tfrac{1}{2}, +\tfrac{1}{2}\right\rangle$$
$$n: \quad |I, I_3\rangle = \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

$$\hat{I}_3 \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} p \\ n \end{pmatrix}$$
$$\hat{I}_+ \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ p \end{pmatrix}$$
$$\hat{I}_- \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} n \\ 0 \end{pmatrix}$$

Electric charge $$Q = I_3 + \frac{1}{2}$$

Chain

$$\begin{array}{lccc} \text{Algebras}: & SU(2) & \supset & SO(2) \\ \text{Generators}: & \hat{I}_\pm, \hat{I}_3 & & \hat{I}_3 \\ \text{Labels}: & |\, I & , & I_3\rangle \end{array}$$

Isospin Symmetry

- Isospin operators form an SU(2) algebra:

$$\left[\hat{I}_3, \hat{I}_\pm\right] = \pm\hat{I}_\pm \ , \qquad \left[\hat{I}_+, \hat{I}_-\right] = 2\hat{I}_3$$

- Assume the nuclear Hamiltonian satisfies

$$\left[H_{\text{nucl}}, \hat{I}_\mu\right] = 0 \ , \qquad \hat{I}_\mu = \sum_{k=1}^{A} \hat{I}_\mu(k)$$

- H_{nucl} has SU(2) symmetry with degenerate states belonging to isobaric multiplets $H_{\text{nucl}} \left|\alpha I I_3\right\rangle = \kappa_0(\alpha, I) \left|\alpha I I_3\right\rangle$

$$I_3 = \frac{1}{2}(Z-N) = -I, -I+1, \ldots, I$$

Isospin Dynamical Symmetry

- Group chain $SU(2) \supset SO(2)$
- The Coulomb interaction breaks the isospin symmetry

$$H = H_{\text{nucl}} + H_{\text{coul}} \approx H_{\text{nucl}} + \kappa_1 \hat{I}_3 + \kappa_2 \hat{I}_3^2$$

$$\left[H, \hat{I}_3\right] = 0, \qquad \left[H, \hat{I}_\pm\right] \neq 0$$

- H has SU(2) dynamical symmetry and SO(2) symmetry
- Degeneracy is lifted according to

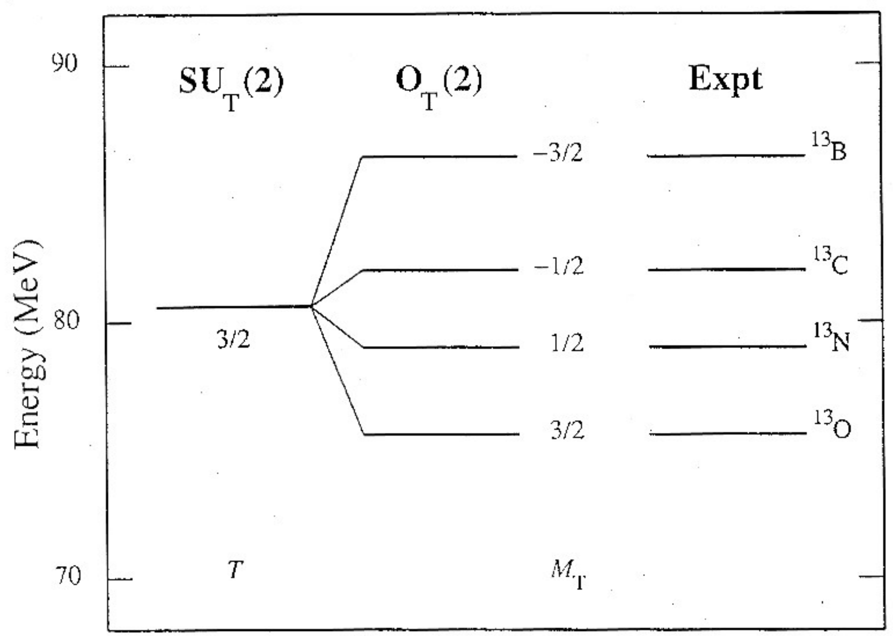
$$\begin{aligned} H \left|\alpha I I_3\right\rangle &= E \left|\alpha I I_3\right\rangle \\ E &= \kappa_0(\alpha, I) + \kappa_1 I_3 + \kappa_2 I_3^2 \end{aligned}$$

Isobaric Multiplet

- Mass equation

$$E(\alpha I I_3) = \kappa_0(\alpha, I) + \kappa_1 I_3 + \kappa_2 I_3^2$$

- Example: I=3/2 multiplet for A=13 nuclei



Z=5, N=8

Z=6, N=7

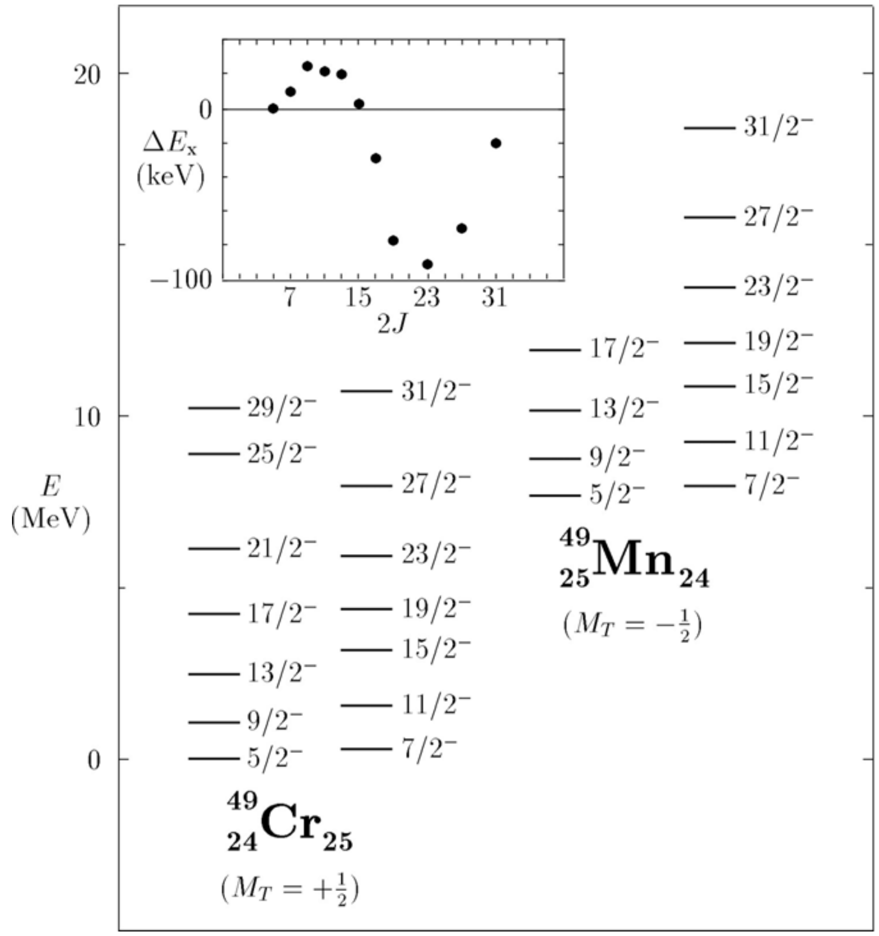
Z=7, N=6

Z=8, N=5

E.P. Wigner, Proc. Welch Found. Conf. (1958) 88

Isospin Symmetry Breaking

- Coulomb displacement energies
- Differences in the relative energies of isobaric analogue states
- Example: I=1/2 doublet of A=49 nuclei.



O'Leary *et al.*, Phys. Rev. Lett. **79** (1997) 4349

Isospin Selection Rules

- SU(2) symmetry implies degenerate isobaric multiplet states $|\alpha I I_3\rangle \;, \quad -I \leq I_3 \leq I$
- Internal E1 transition operator is isovector

$$T_\mu^{(E1)} = \sum_{k=1}^{A} e_k r_\mu(k) = \underbrace{\frac{e}{2}\sum_{k=1}^{A} r_\mu(k)}_{\text{cm motion}} + \underbrace{e\sum_{k=1}^{A} i_3(k) r_\mu(k)}_{\text{isovector}}$$

- Selection rule for N=Z (I_3=0) nuclei:

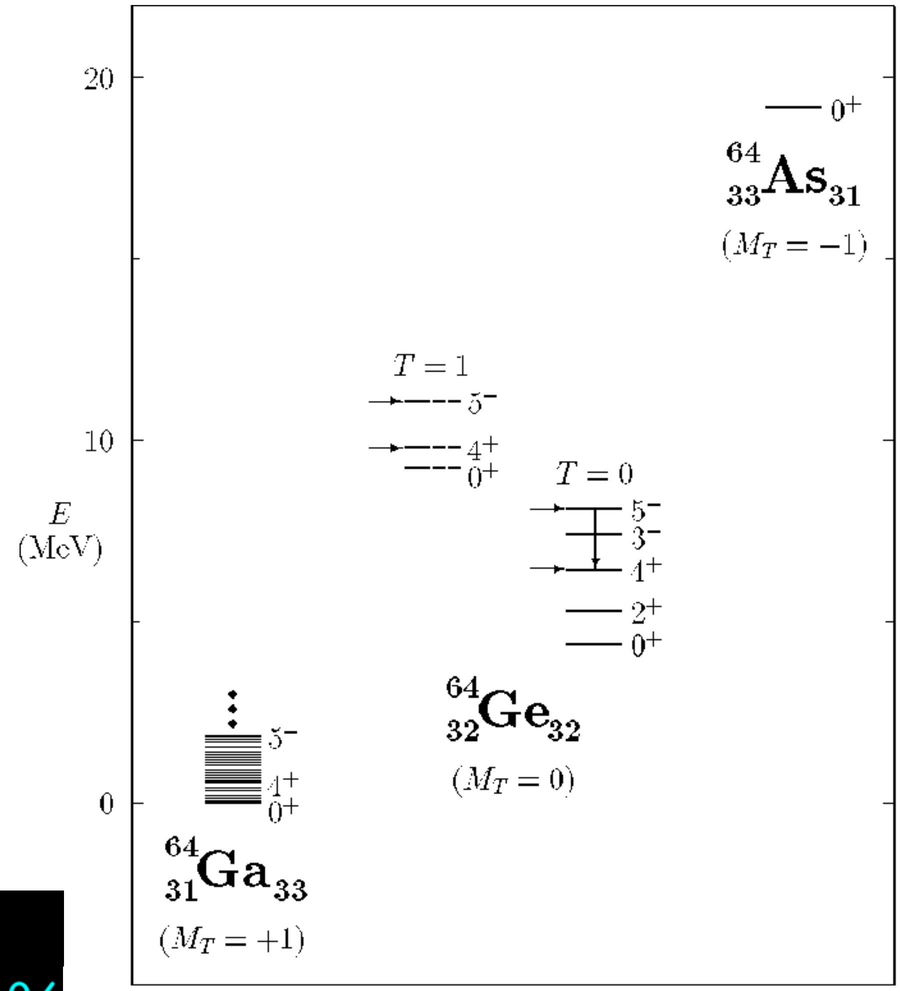
E1 transitions are forbidden between states with the same isospin

L.E.H. Trainor, Phys. Rev. **85** (1952) 962
L.A. Radicati, Phys. Rev. **87** (1952) 521

E1 Transitions and Isospin Mixing

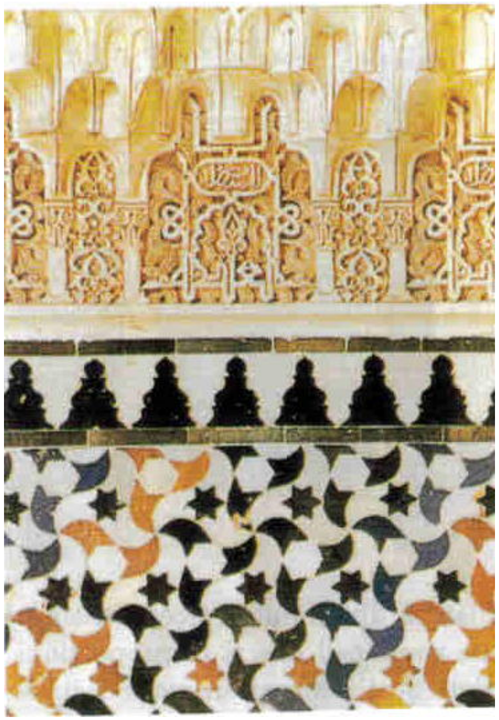
- $B(E1;5^- \to 4^+)$ in ^{64}Ge from:
  - Lifetime of 5^- level.
  - $\delta(E1/M2)$ mixing ratio of $5^- \to 4^+$ transition.
  - Relative intensities of transitions de-exciting 5^-.
- Estimate of isospin mixing:

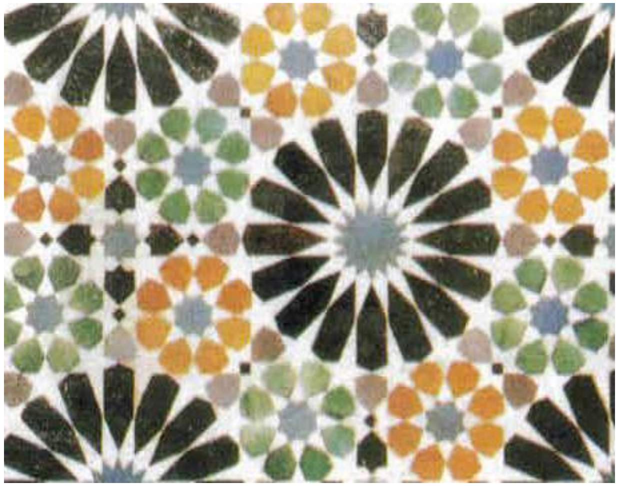
$$P(J^P = 5^-, I = 1) \approx P(J^P = 4^+, I = 1) \approx 2.5\ \%$$

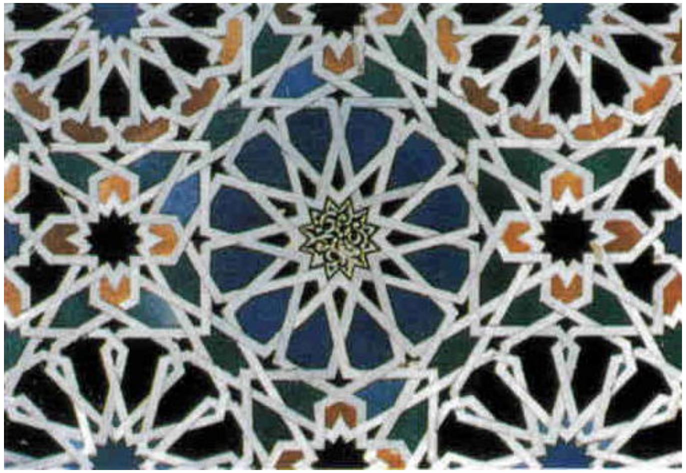


E.Farnea *et al.*, Phys. Lett. B **551** (2003) 56

Alhambra, Spain







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- Wigner – Group theory and its applications to the quantum mechanics of atomic spectra (1931)
- Hamermesh – Group theory and its application to physical problems (1962)
- Wybourne – Classical groups for physicists (1974)
- Elliott and Dawber – Symmetry in physics (1979)
- Van Isacker – Rep. Prog. Phys. 62, 1661-1717 (1999)
- Iachello – Lie algebras and applications (2006)

General Interest

- Weyl – Symmetry (1952)
- Hargittai and Hargittai – Symmetry, a unifying concept (1994)
- Abas and Salman – Symmetries of islamic geometric patterns (1995)
- Lederman and Hill – Symmetry and the beautiful universe (2004)