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Nuclear Supersymmetry

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Abstract. The concept of symmetries in physics is briefly reviewed. In the first part of these lecture notes, some of the basic mathematical tools needed for the understanding of symmetries in nature are presented, namely group theory, Lie groups and Lie algebras, and Noether's theorem. In the second part, some applications of symmetries in physics are discussed, ranging from isospin and flavor symmetry to more recent developments involving the interacting boson model and its extension to supersymmetries in nuclear physics.

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INTRODUCTION

Nuclear structure physics has seen an impressive progress in the development of *ab initio* methods (no-core shell model, Green's Function Monte Carlo, Coupled Clusters, ...), mean-field techniques and effective field theories for which the ultimate goal is *an exact treatment of nuclei utilizing the fundamental interactions between nucleons* [1]. All involve large scale calculations and therefore rely heavily on the available computing power and the development of efficient algorithms to obtain the desired results.

A different, complementary, approach is that of symmetries and algebraic methods. Rather than trying to solve the complex nuclear many-body problem numerically, the aim is to identify effective degrees of freedom, develop schematic models based upon these degrees of freedom and study their solutions by means of symmetries, etc. Aside from their esthetic appeal, symmetries provide energy formula, selection rules and closed expressions for electromagnetic transition rates and transfer strengths which can be used as benchmarks to study and interpret the experimental data, even if these symmetries may be valid only approximately. Historically, symmetries have played an important role in nuclear physics. Examples are isospin symmetry, the Wigner supermultiplet theory, special solutions to the Bohr Hamiltonian, the Elliott model, pseudo-spin symmetries and the dynamical symmetries and supersymmetries of the IBM and its extensions.

The purpose of these lecture notes is to discuss several new developments in nuclear supersymmetry, in particular evidence for the existence of a new supersymmetric quartet in the $A \sim 190$ mass region, consisting of the ^{192,193}Os and ^{193,194}Ir nuclei, and correlations between different one- and two-nucleon transfer reactions. In the first part of these lecture notes, a brief review is given on some of the basic mathematical concepts needed for the understanding of symmetries in nature, namely that of group theory, Lie groups and Lie algebras, and Noether's theorem. In the second part, these ideas are illustrated by some applications in physics, ranging from isospin and flavor symmetry to more recent

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developments involving the interacting boson model and its extension to supersymmetries in nuclear physics. Some recent review articles on the concept of symmetries in physics are [2, 3, 4, 5].

SYMMETRIES AND GROUP THEORY

Symmetry and its mathematical framework—group theory—play an increasingly important role in physics. Both classical and quantum systems usually display great complexity, but the analysis of their symmetry properties often gives rise to simplifications and new insights which can lead to a deeper understanding. In addition, symmetries themselves can point the way toward the formulation of a correct physical theory by providing constraints and guidelines in an otherwise intractable situation. It is remarkable that, in spite of the wide variety of systems one may consider, all the way from classical ones to molecules, nuclei, and elementary particles, group theory applies the same basic principles and extracts the same kind of useful information from all of them. This universality in the applicability of symmetry considerations is one of the most attractive features of group theory. Most people have an intuitive understanding of symmetry, particularly in its most obvious manifestation in terms of geometric transformations that leave a body or system invariant. This interpretation, however, is not enough to readily grasp its deep connections with physics, and it thus becomes necessary to generalize the notion of symmetry transformations to encompass more abstract ideas. The mathematical theory of these transformations is the subject matter of group theory.

Group theory was developed in the beginning of the 19th century by Evariste Galois who pointed out the relation between the existence of algebraic solutions of a polynomial equation and the group of permutations associated with the equation. Another important contribution was made in the 1870's by Sophus Lie who studied the mathematical theory of continuous transformations which led to the introduction of the basic concepts and operations of what are now known as Lie groups and Lie algebras. The deep connection between the abstract world of symmetries and dynamics—forces and motion and the fundamental laws of nature—was elucidated by Emmy Noether in the early 20th century.

The concept of symmetry has played a major role in physics, especially in the 20th century with the development of quantum mechanics and quantum field theory. There is an enormously wide range of applications of symmetries in physics. Some of the most important ones are listed below [2].

- *Geometric symmetries* describe the arrangement of constituent particles into a geometric structure, for example the atoms in a molecule.
- *Permutation symmetries* in quantum mechanics lead to Fermi-Dirac and Bose-Einstein statistics for a system of identical particles with half-integer spin (fermions) and integer spin (bosons), respectively.
- *Space-time symmetries* fix the form of the equations governing the motion of the constituent particles. For example, the form of the Dirac equation for a relativistic spin-1/2 particle

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0$$

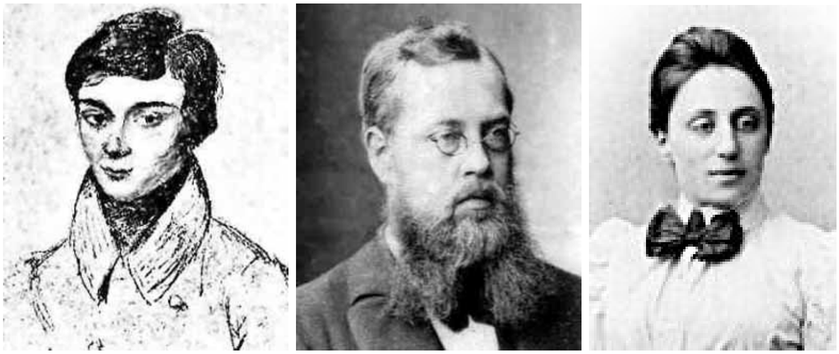

FIGURE 1. Evariste Galois (1811-1832), Sophus Lie (1842-1899) and Emmy Noether (1882-1935) [6].

is determined by Lorentz invariance.

- *Gauge symmetries* fix the form of the interaction between constituent particles and external fields. For example, the form of the Dirac equation for a relativistic spin-1/2 particle in an external electromagnetic field A_μ

$$\left[\gamma^\mu(i\partial_\mu - eA_\mu) - m\right]\psi(x) = 0$$

  is dictated by the gauge symmetry of the electromagnetic interaction. The (electro-)weak and strong interactions are also governed by gauge symmetries.
- *Dynamical symmetries* fix the form of the interactions between constituent particles and/or external fields and determine the spectral properties of quantum systems. An early example was discussed by Pauli in 1926 [7] who recognized that the Hamiltonian of a particle in a Coulomb potential is invariant under four-dimensional rotations generated by the angular momentum and the Runge-Lenz vector.

ELEMENTS OF GROUP THEORY

In this section, some general properties of group theory are reviewed. For a more thorough discussion of the basic concepts and its properties, the reader is referred to the literature [8, 9, 10, 11, 12, 13, 14].

Definition of a group

The concept of a group was introduced by Galois in a study of the existence of algebraic solutions of a polynomial equations. An abstract group G is defined by a set of elements $(\hat{G}_i, \hat{G}_j, \hat{G}_k, \ldots)$ for which a "multiplication" rule (indicated here by $\circ$)

combining these elements exists and which satisfies the following conditions.

- *Closure.*
  If $\hat{G}_i$ and $\hat{G}_j$ are elements of the set, so is their product $\hat{G}_i \circ \hat{G}_j$.
- *Associativity.*
  The following property is always valid:
  $$\hat{G}_i \circ (\hat{G}_j \circ \hat{G}_k) = (\hat{G}_i \circ \hat{G}_j) \circ \hat{G}_k \ .$$
- *Identity.*
  There exists an element $\hat{E}$ of G satisfying
  $$\hat{E} \circ \hat{G}_i = \hat{G}_i \circ \hat{E} = \hat{G}_i \ .$$
- *Inverse.*
  For every $\hat{G}_i$ there exists an element $\hat{G}_i^{-1}$ such that
  $$\hat{G}_i \circ \hat{G}_i^{-1} = \hat{G}_i^{-1} \circ \hat{G}_i = \hat{E} \ .$$

The number of elements is called the *order* of the group. If in addition the elements of a group satisfy the condition of commutativity, the group is called an Abelian group.

- *Commutativity.*
  All elements obey
  $$\hat{G}_i \circ \hat{G}_j = \hat{G}_j \circ \hat{G}_i \ .$$

Lie groups and Lie algebras

For continuous (or Lie) groups all elements may be obtained by exponentiation in terms of a basic set of elements $\hat{g}_i$, $i = 1, 2, \ldots, s$, called *generators*, which together form the *Lie algebra* associated with the Lie group. A simple example is provided by the group of rotations in two-dimensional space, with elements that may be realized as

$$\hat{G}(\alpha) = \exp[-i\alpha\hat{l}_z] \ , \tag{1}$$

where α is the angle of rotation and

$$\hat{l}_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \ , \tag{2}$$

is the generator of these transformations in the x–y plane. Three-dimensional rotations require the introduction of two additional generators, associated with rotations in the z–x and y–z planes,

$$\hat{l}_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \ , \qquad \hat{l}_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \ , \tag{3}$$

Finite rotations can then be parametrized by three angles (which may be chosen to be the Euler angles) and expressed as a product of exponentials of the generators of Eqs. (2) and (3). Evaluating the commutators of these operators, we find

$$[\hat{l}_x, \hat{l}_y] = i\hat{l}_z \;, \qquad [\hat{l}_y, \hat{l}_z] = i\hat{l}_x \;, \qquad [\hat{l}_z, \hat{l}_x] = i\hat{l}_y \;, \tag{4}$$

which illustrates the closure property of the group generators. In general, the operators $\hat{g}_i$, $i = 1, 2, \ldots, s$, define a *Lie algebra* if they close under commutation

$$[\hat{g}_i, \hat{g}_j] = \sum_k c_{ij}^k \hat{g}_k \;, \tag{5}$$

and satisfy the Jacobi identity

$$[\hat{g}_i, [\hat{g}_j, \hat{g}_k]] + [\hat{g}_k, [\hat{g}_i, \hat{g}_j]] + [\hat{g}_j, [\hat{g}_k, \hat{g}_i]] = 0 \;. \tag{6}$$

The constants c_{ij}^k are called *structure constants*, and determine the properties of both the Lie algebra and its associated Lie group. Lie groups have been classified by Cartan, and many of their properties have been established.

The group of unitary transformations in n dimensions is denoted by $U(n)$ and of rotations in n dimensions by $SO(n)$ (Special Orthogonal). The corresponding Lie algebras are sometimes indicated by lower case symbols, $u(n)$ and $so(n)$, respectively.

Symmetries and conservation laws

Symmetry in physics is expressed by the invariance of a Lagrangian or of a Hamiltonian or, equivalently, of the equations of motion, with respect to some group of transformations. The connection between the abstract concept of symmetries and dynamics is formulated as Noether's theorem which says that, irrespective of a classical or a quantum mechanical treatment, an invariant Lagrangian or Hamiltonian with respect to a continuous symmetry implies a set of conservation laws [15]. For example, the conservation of energy, momentum and angular momentum are a consequence of the invariance of the system under time translations, space translations and rotations, respectively.

In quantum mechanics, continuous symmetry transformations can in general be expressed as

$$U = \exp\left(i \sum_j \alpha_j \hat{g}_j \right) \;. \tag{7}$$

States and operators transform as

$$|\psi\rangle \to |\psi'\rangle = U |\psi\rangle \;, \qquad A \to A' = UAU^\dagger \;. \tag{8}$$

For the Hamiltonian one then has

$$H \to H' = UHU^\dagger = H + i \sum_j \alpha_j [\hat{g}_j, H] + \mathcal{O}(\alpha^2) \;. \tag{9}$$

When the physical system is invariant under the symmetry transformations U, the Hamiltonian remains the same $H' = H$. Therefore, the Hamiltonian commutes with the generators of the symmetry transformation

$$\left[\hat{g}_j, H\right] = 0 \,, \tag{10}$$

which implies that the generators are constants of the motion. Eq. (10), together with the closure relation of the generators of Eq. (5), constitutes the definition of the symmetry algebra for a time-independent system.

Constants of motion and state labeling

For any Lie algebra one may construct one or more operators $\hat{\mathscr{C}}_l$ which commute with all the generators $\hat{g}_j$

$$[\hat{\mathscr{C}}_l, \hat{g}_j] = 0, \qquad l = 1,2,\ldots,r, \quad j = 1,2,\ldots,s \,. \tag{11}$$

These operators are called *Casimir operators* or *Casimir invariants*. They may be linear, quadratic, or higher order in the generators. The number r of linearly independent Casimir operators is called the rank of the algebra [11]. This number coincides with the maximum subset of generators which commute among themselves (called *weight generators*)

$$[\hat{g}_\alpha, \hat{g}_\beta] = 0 \,, \qquad \alpha, \beta = 1,2,\ldots,r \,, \tag{12}$$

where greek labels were used to indicate that they belong to the subset satisfying Eq. (12). The operators $(\hat{\mathscr{C}}_l,\, \hat{g}_\alpha)$ may be simultaneously diagonalized and their eigenvalues used to label the corresponding eigenstates.

To illustrate these definitions, we consider the $su(2)$ algebra $(\hat{j}_x,\, \hat{j}_y,\, \hat{j}_z)$ with commutation relations

$$[\hat{j}_x, \hat{j}_y] = i\hat{j}_z \,, \qquad [\hat{j}_z, \hat{j}_x] = i\hat{j}_y \,, \qquad [\hat{j}_y, \hat{j}_z] = i\hat{j}_x \,, \tag{13}$$

which is isomorphic to the $so(3)$ commutators given in Eq. (4). From Eq. (13) one can conclude that the rank of the algebra is $r = 1$. Therefore one can choose $\hat{j}_z$ as the generator to diagonalize together with the Casimir invariant

$$\hat{j}^2 = \hat{j}_x^2 + \hat{j}_y^2 + \hat{j}_z^2 \,. \tag{14}$$

The eigenvalues and branching rules for the commuting set $(\hat{\mathscr{C}}_l,\, \hat{g}_\alpha)$ can be determined solely from the commutation relations Eq. (5). In the case of $su(2)$ the eigenvalue equations are

$$\hat{j}^2|jm\rangle = n_j|jm\rangle \,, \qquad \hat{j}_z|jm\rangle = m|jm\rangle \,, \tag{15}$$

where j is an index to distinguish the different $\hat{j}^2$ eigenvalues. Defining the raising and lowering operators

$$\hat{j}_\pm = \hat{j}_x \pm i\hat{j}_y \,, \tag{16}$$

and using Eq. (13), one finds the well-known results

$$n_j = j(j+1) \;, \qquad j = 0, \frac{1}{2}, 1, \ldots, \tag{17}$$

with $m = -j, -j+1, \ldots, j$. As a bonus, the action of $\hat{j}_\pm$ on the $|jm\rangle$ eigenstates is determined to be

$$\hat{j}_\pm |jm\rangle = \sqrt{(j \mp m)(j \pm m+1)}\, |jm \pm 1\rangle \;. \tag{18}$$

In the case of a general Lie algebra, see Eq. (5), this procedure becomes quite complicated, but it requires the same basic steps. The analysis leads to the algebraic determination of eigenvalues, branching rules, and matrix elements of raising and lowering operators [11].

The *symmetry* algebra provides constants of the motion, which in turn lead to quantum numbers that label the states associated with a given energy eigenvalue. The raising and lowering operators in this algebra only connect degenerate states. The dynamical algebra, however, defines the whole set of eigenstates associated with a given system. The generators are no longer constants of the motion as not all commute with the Hamiltonian. The raising and lowering operators may now connect all states with each other.

Dynamical symmetries

In this section we show how the concepts presented in the previous sections lead to an algebraic approach which can be applied to the study of different physical systems. We start by considering again Eq. (10) which describes the invariance of a Hamiltonian under the algebra $g \equiv (\hat{g}_j)$

$$[H, \hat{g}_j] = 0 \;, \tag{19}$$

implying that g plays the role of symmetry algebra for the system. An eigenstate of H with energy E may be written as $|\Gamma\gamma\rangle$, where Γ labels the irreducible representations of the group G corresponding to g and γ distinguishes between the different eigenstates with energy E (and may be chosen to correspond to irreducible representations of subgroups of G). The energy eigenvalues of the Hamiltonian in Eq. (19) thus depend only on Γ

$$H|\Gamma\gamma\rangle = E(\Gamma)|\Gamma\gamma\rangle \;. \tag{20}$$

The generators $\hat{g}_j$ do not admix states with different Γ's.

Let's now consider the chain of algebras

$$g_1 \supset g_2 \;, \tag{21}$$

which will lead to the introduction of the concept of *dynamical symmetry*. Here g_2 is a subalgebra of g_1, $g_2 \subset g_1$, *i.e.* its generators form a subset of the generators of g_1 and close under commutation. If g_1 is a symmetry algebra for H, its eigenstates can be labeled as $|\Gamma_1\gamma_1\rangle$. Since $g_2 \subset g_1$, g_2 must also be a symmetry algebra for H and, consequently, its eigenstates labeled as $|\Gamma_2\gamma_2\rangle$. Combination of the two properties leads to the eigenequation

$$H|\Gamma_1\Gamma_2\gamma_2\rangle = E(\Gamma_1)|\Gamma_1\Gamma_2\gamma_2\rangle \ , \tag{22}$$

where the role of γ_1 is played by $\Gamma_2\gamma_2$ and hence the eigenvalues depend only on Γ_1. This process may be continued when there are further subalgebras, that is, $g_1 \supset g_2 \supset g_3 \supset \cdots$, in which case γ_2 is substituted by $\Gamma_3\gamma_3$, and so on.

In many physical applications the original assumption that g_1 is a symmetry algebra of the Hamiltonian is found to be too strong and must be relaxed, that is, one is led to consider the breaking of this symmetry. An elegant way to do so is by considering a Hamiltonian of the form

$$H' = a\hat{\mathscr{C}}_{l_1}(g_1) + b\hat{\mathscr{C}}_{l_2}(g_2) \ , \tag{23}$$

where $\hat{\mathscr{C}}_{l_i}(g_i)$ is a Casimir invariant of g_i. Since $[H', \hat{g}_i] = 0$ for $\hat{g}_i \in g_2$, H' is invariant under g_2, but not anymore under g_1 because $[\hat{\mathscr{C}}_{l_2}(g_2), \hat{g}_i] \neq 0$ for $\hat{g}_i \notin g_2$. The new *symmetry algebra* is thus g_2 while g_1 now plays the role of *dynamical algebra* for the system, as long as all states we wish to describe are those originally associated with $E(\Gamma_1)$. The extent of the symmetry breaking depends on the ratio b/a. Furthermore, since H' is given as a combination of Casimir operators, its eigenvalues can be obtained in closed form

$$H'|\Gamma_1\Gamma_2\gamma_2\rangle = [aE_{l_1}(\Gamma_1) + bE_{l_2}(\Gamma_2)]\,|\Gamma_1\Gamma_2\gamma_2\rangle \ . \tag{24}$$

The kind of symmetry breaking caused by interactions of the form (23) is known as *dynamical-symmetry breaking* and the remaining symmetry is called a *dynamical symmetry* of the Hamiltonian H'. From Eq. (24) one concludes that even if H' is not invariant under g_1, its eigenstates are the same as those of $\hat{H}$ in Eq. (22). The dynamical-symmetry breaking thus splits but does not admix the eigenstates.

In the last part of these lecture notes, we discuss some applications of the algebraic approach in nuclear and particle physics. The algebraic approach, both in the sense we have defined here and in its generalizations to other fields of research, has become an important tool in the search for a unified description of physical phenomena.

ISOSPIN SYMMETRY

Some of these ideas can be illustrated with well-known examples. In 1932 Heisenberg considered the occurrence of isospin multiplets in nuclei [16]. To a good approximation, the strong interaction between nucleons does not distinguish between protons and neutrons. In the isospin formalism, the proton and neutron are treated as one and the same particle: the nucleon with isospin $t = \frac{1}{2}$. The isospin projections $m_t = +\frac{1}{2}$ and $-\frac{1}{2}$ are identified with the proton and the neutron, respectively. The total isospin of the nucleus is denoted by T and its projection by M_T. In the notation used above (without making the distinction between algebras and groups), G_1 is in this case the isospin group $SU_T(2)$ generated by the operators $\hat{T}_x$, $\hat{T}_y$, and $\hat{T}_z$ which satisfy commutation relations of Eq. (13), and G_2 can be identified with $SO_T(2)$ generated by $\hat{T}_z$). An isospin-invariant Hamiltonian commutes with $\hat{T}_x$, $\hat{T}_y$, and $\hat{T}_z$, and hence the eigenstates $|TM_T\rangle$ with fixed T and

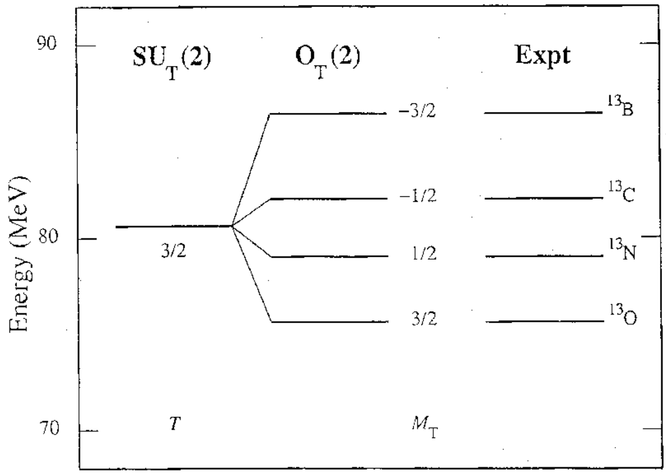

FIGURE 2. Binding energies of the $T = 3/2$ isobaric analog states with angular momentum and parity $J^\pi = 1/2^-$ in ^{13}B, ^{13}C, ^{13}N, and ^{13}O [4]. The column on the left is obtained for an exact $SU_T(2)$ *symmetry*, which predicts states with different M_T to be degenerate. The middle column is obtained in the case of an $SU_T(2)$ *dynamical symmetry*, Eq. (25) with parameters $a = 80.59$, $b = -2.96$, and $c = -0.26$ MeV.

$M_T = -T, -T+1, \ldots, T$ are degenerate in energy. However, the electromagnetic interaction breaks isospin invariance due to difference in electric charge of the proton and the neutron, and lifts the degeneracy of the states $|TM_T\rangle$. It is assumed that this symmetry breaking occurs dynamically, and since the Coulomb force has a two-body character, the breaking terms are at most quadratic in $\hat{T}_z$ [12]. The energies of the corresponding nuclear states with the same T are then given by

$$E(M_T) = a + bM_T + cM_T^2 \ , \tag{25}$$

and $SU_T(2)$ becomes the dynamical symmetry for the system while $SO_T(2)$ is the symmetry algebra. The dynamical symmetry breaking thus implied that the eigenstates of the nuclear Hamiltonian have well-defined values of T and M_T. Extensive tests have shown that indeed this is the case to a good approximation, at least at low excitation energies and in light nuclei [17]. Eq. (25) can be tested in a number of cases. In Fig. 2 a $T = 3/2$ multiplet consisting of states in the nuclei ^{13}B, ^{13}C, ^{13}N, and ^{13}O is compared with the theoretical prediction of Eq. (25).

FLAVOR SYMMETRY

A less trivial example of dynamical-symmetry breaking is provided by the Gell-Mann–Okubo mass-splitting formula for elementary particles [18, 19]. In the previous example, we saw that the near equality of the neutron and proton masses suggested the existence

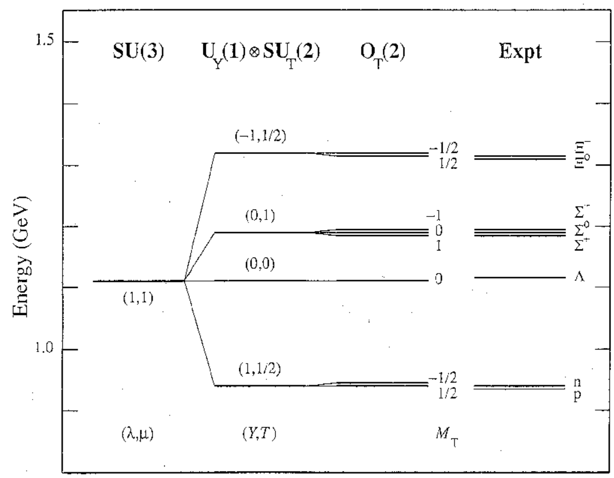

FIGURE 3. Mass spectrum of the ground state baryon octet [4]. The column on the left is obtained for an exact $SU(3)$ symmetry, which predicts all masses to be the same, while the next two columns represent successive breakings of this symmetry in a dynamical manner. The column under $SO_T(2)$ is obtained with Eq. (29) with parameters $a = 1111.3$, $b = -189.6$, $d = -39.9$, $e = -3.8$, and $f = 0.9$ MeV.

of isospin multiplets which was later confirmed at higher energies for other particles. Gell-Mann and Ne'eman proposed independently a dynamical algebra to further classify and order these different isospin multiplets of hadrons in terms of $SU(3)$ representations [20] . Baryons were found to occur in decuplets, octets and singlets, whereas mesons appear only in octets and singlets. The members of a $SU(3)$ multiplet are labeled by their isospin T, M_T and hypercharge Y quantum numbers, according to the group chain

$$\begin{array}{ccccccccc} SU(3) & \supset & U_Y(1) & \otimes & SU_T(2) & \supset & U_Y(1) & \otimes & SO_T(2) \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ (\lambda,\mu) & & Y & & T & & & & M_T \end{array} \tag{26}$$

If one would assume $SU(3)$ invariance, all particles in a multiplet would have the same mass, but since the experimental masses of other baryons differ from the nucleon masses by hundreds of MeV, the $SU(3)$ symmetry clearly must be broken.

Dynamical symmetry breaking allows the baryon states to still be classified by Eq. (26). Following the procedure outlined above and keeping up to quadratic terms, one finds a mass operator of the form

$$\begin{aligned} \hat{M} &= a + b\hat{\mathscr{C}}_{1U_Y(1)} + c\hat{\mathscr{C}}^2_{1U_Y(1)} + d\hat{\mathscr{C}}_{2SU_T(2)} \\ &\quad + e\hat{\mathscr{C}}_{1SO_T(2)} + f\hat{\mathscr{C}}^2_{1SO_T(2)} , \end{aligned} \tag{27}$$

with eigenvalues

$$M(Y,T,M_T) \quad = \quad a + bY + cY^2 + dT(T+1)$$

$$+e M_T + f M_T^2 \; . \tag{28}$$

A further assumption regarding the $SU(3)$ tensor character of the strong interaction [18, 19] leads to a relation between the coefficients c and d in Eq. (28), $c = -d/4$

$$\begin{aligned} M'(Y,T,M_T) &= a + bY + d\left[T(T+1) - \frac{1}{4}Y^2\right] \\ &\quad + e M_T + f M_T^2 \; . \end{aligned} \tag{29}$$

If one neglects the isospin breaking due to the last two terms, one recovers the Gell-Mann-Okubo mass formula. In Fig. 3 this process of successive dynamical-symmetry breaking is illustrated with the octet representation containing the neutron and the proton and the Λ, Σ, and Ξ baryons.

NUCLEAR SUPERSYMMETRY

Nuclear supersymmetry (n-SUSY) is a composite-particle phenomenon, linking the properties of bosonic and fermionic systems, framed in the context of the Interacting Boson Model of nuclear structure [21]. Composite particles, such as the α-particle are known to behave as approximate bosons. As He atoms they become superfluid at low temperatures, an under certain conditions can also form Bose-Einstein condensates. At higher densities (or temperatures) the constituent fermions begin to be felt and the Pauli principle sets in. Odd-particle composite systems, on the other hand, behave as approximate fermions, which in the case of the Interacting Boson-Fermion Model are treated as a combination of bosons and an (ideal) fermion [22]. In contrast to the theoretical construct of supersymmetric particle physics, where SUSY is postulated as a generalization of the Lorentz-Poincare invariance at a fundamental level, experimental evidence has been found for n-SUSY [23, 24, 25, 26, 27, 28, 29] as we shall discuss below. Nuclear supersymmetry should not be confused with fundamental SUSY, which predicts the existence of supersymmetric particles, such as the photino and the selectron for which, up to now, no evidence has been found. If such particles exist, however, SUSY must be strongly broken, since large mass differences must exist among superpartners, or otherwise they would have been already detected. Nuclear supersymmetry, on the other hand, is a theory that establishes precise links among the spectroscopic properties of certain neighboring nuclei. Even-even and odd-odd nuclei are composite bosonic systems, while odd-A nuclei are fermionic. It is in this context that n-SUSY provides a theoretical framework where bosonic and fermionic systems are treated as members of the same supermultiplet [25]. Nuclear supersymmetry treats the excitation spectra and transition intensities of the different nuclei as arising from a single Hamiltonian and a single set of transition operators. Nuclear supersymmetry was originally postulated as a symmetry among pairs of nuclei [23, 24, 25], and was subsequently extended to quartets of nuclei, where odd-odd nuclei could be incorporated in a natural way [30]. Evidence for the existence of n-SUSY (albeit possibly significantly broken) grew over the years, specially for the quartet of nuclei ^{194}Pt, ^{195}Au, ^{195}Pt and ^{196}Au, but only recently more systematic evidence was found [27, 28, 29].

We first present a pedagogic review of dynamical (super)symmetries in even- and odd-mass nuclei, which is based in part on [26]. Next we discuss the generalization of these concepts to include the neutron-proton degree of freedom.

Dynamical symmetries in even-even nuclei

Dynamical supersymmetries were introduced in nuclear physics in 1980 by Franco Iachello in the context of the Interacting Boson Model (IBM) and its extensions [23]. The spectroscopy of atomic nuclei is characterized by the interplay between collective (bosonic) and single-particle (fermionic) degrees of freedom.

The IBM describes collective excitations in even-even nuclei in terms of a system of interacting monopole and quadrupole bosons with angular momentum $l=0,2$ [21]. The bosons are associated with the number of correlated proton and neutron pairs, and hence the number of bosons N is half the number of valence nucleons. Since it is convenient to express the Hamiltonian and other operators of interest in second quantized form, we introduce creation, $s^\dagger$ and $d_m^\dagger$, and annihilation, s and d_m, operators, which altogether can be denoted by $b_i^\dagger$ and b_i with $i=l,m$ ($l=0,2$ and $-l \leq m \leq l$). The operators $b_i^\dagger$ and b_i satisfy the commutation relations

$$[b_i, b_j^\dagger] \;=\; \delta_{ij} \;, \qquad [b_i^\dagger, b_j^\dagger] \;=\; [b_i, b_j] \;=\; 0 \;. \tag{30}$$

The bilinear products

$$B_{ij} \;=\; b_i^\dagger b_j \;, \tag{31}$$

generate the algebra of $U(6)$ the unitary group in 6 dimensions

$$[B_{ij}, B_{kl}] \;=\; B_{il}\,\delta_{jk} - B_{kj}\,\delta_{il} \;. \tag{32}$$

We want to construct states and operators that transform according to irreducible representations of the rotation group (since the problem is rotationally invariant). The creation operators $b_i^\dagger$ transform by definition as irreducible tensors under rotation. However, the annihilation operators b_i do not. It is an easy exercise to contruct operators that do transform appropriately

$$\tilde{b}_{lm} \;=\; (-)^{l-m} b_{l,-m} \;. \tag{33}$$

The 36 generators of Eq. (31) can be rewritten in angular-momentum-coupled form as

$$(b_l^\dagger \tilde{b}_{l'})^{(\lambda)}_\mu \;=\; \sum_{mm'} \langle l,m,l',m'|\lambda,\mu\rangle \, b_{lm}^\dagger \tilde{b}_{l'm'} \;. \tag{34}$$

The one- and two-body Hamiltonian can be expressed in terms of the generators of $U(6)$ as

$$H \;=\; \sum_l \varepsilon_l \sum_m b_{lm}^\dagger b_{lm} + \sum_\lambda \sum_{l_1 l_2 l_3 l_4} u^{(\lambda)}_{l_1 l_2 l_3 l_4} \left[(b_{l_1}^\dagger \tilde{b}_{l_2})^{(\lambda)} \cdot (b_{l_3}^\dagger \tilde{b}_{l_4})^{(\lambda)} + h.c. \right] \;. \tag{35}$$

In general, the Hamiltonian has to be diagonalized numerically to obtain the energy eigenvalues and wave functions. There exist, however, special situations in which the eigenvalues can be obtained in closed, analytic form. These special solutions provide a framework in which energy spectra and other nuclear properties (such as quadrupole transitions and moments) can be interpreted in a qualitative way. These situations correspond to dynamical symmetries of the Hamiltonian [21].

The concept of dynamical symmetry has been shown to be a very useful tool in different branches of physics. A well-known example in nuclear physics is the Elliott $SU(3)$ model [31] to describe the properties of light nuclei in the *sd* shell. Another example is the $SU(3)$ flavor symmetry of Gell-Mann and Ne'eman [20] to classify the baryons and mesons into flavor octets, decuplets and singlets and to describe their masses with the Gell-Mann-Okubo mass formula, as described in the previous sections.

The group structure of the IBM Hamiltonian is that of $G = U(6)$. Since nuclear states have good angular momentum, the rotation group in three dimensions $SO(3)$ should be included in all subgroup chains of G [21]

$$U(6) \supset \begin{cases} U(5) \supset SO(5) \supset SO(3) \\ \\ SU(3) \supset SO(3) \\ \\ SO(6) \supset SO(5) \supset SO(3) \end{cases} \tag{36}$$

The three dynamical symmetries which correspond to the group chains in Eq. (36) are limiting cases of the IBM and are usually referred to as the $U(5)$ (vibrator), the $SU(3)$ (axially symmetric rotor) and the $SO(6)$ (γ-unstable rotor).

Here we consider a simplified form of the general expression of the IBM Hamiltonian of Eq. (35) that contains the main features of collective motion in nuclei

$$H \;=\; \varepsilon\, \hat{n}_d - \kappa \hat{Q}(\chi) \cdot \hat{Q}(\chi) \;, \tag{37}$$

where n_d counts the number of quadrupole bosons

$$\hat{n}_d \;=\; \sqrt{5}\,(d^\dagger \tilde{d})^{(0)} \;=\; \sum_m d_m^\dagger d_m \;, \tag{38}$$

and Q is the quadrupole operator

$$\hat{Q}(\chi) \;=\; (s^\dagger \tilde{d} + d^\dagger \tilde{s})^{(2)} + \chi\, (d^\dagger \tilde{d})^{(2)} \;. \tag{39}$$

The three dynamical symmetries are recovered for different choices of the coefficients ε, κ and χ. Since the IBM Hamiltonian conserves the number of bosons and is invariant under rotations, its eigenstates can be labeled by the total number of bosons N and the angular momentum L.

The $U(5)$ limit

In the absence of a quadrupole-quadrupole interaction $\kappa = 0$, the Hamiltonian of Eq. (37) becomes proportional to the linear Casimir operator of $U(5)$

$$H_1 = \varepsilon \hat{n}_d = \varepsilon \hat{\mathscr{C}}_{1U(5)} . \tag{40}$$

In addition to N and L, the basis states can be labeled by the quantum numbers n_d and τ, which characterize the irreducible representations of $U(5)$ and $SO(5)$. Here n_d represents the number of quadrupole bosons and τ the boson seniority. The eigenvalues of H_1 are given by the expectation value of the Casimir operator

$$E_1 = \varepsilon n_d . \tag{41}$$

In this case, the energy spectrum is characterized by a series of multiplets, labeled by the number of quadrupole bosons, at a constant energy spacing which is typical for a vibrational nucleus (see Fig. 4).

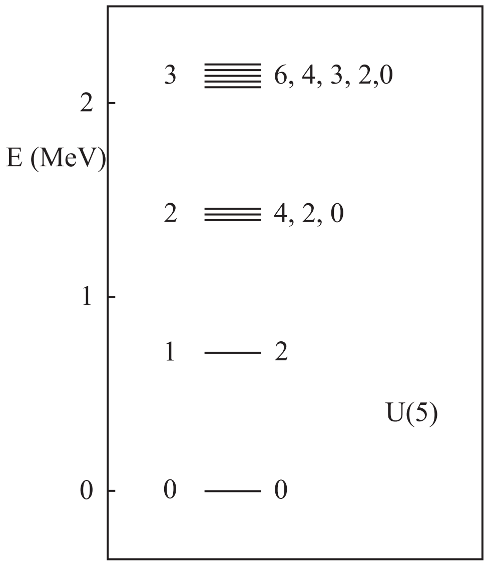

FIGURE 4. Schematic energy spectrum of an even-even nucleus with $U(5)$ symmetry and $N = 3$. The number of quadrupole bosons n_d is shown on the left and the angular momentum L belonging to each oscillator multiplet on the right.

The $SU(3)$ limit

For the quadrupole-quadrupole interaction, we can distinguish two situations in which the eigenvalue problem can be solved analytically. If $\chi = \mp\sqrt{7}/2$, the Hamiltonian has a $SU(3)$ dynamical symmetry

$$H_2 = -\kappa \hat{Q}(\mp\sqrt{7}/2) \cdot \hat{Q}(\mp\sqrt{7}/2) = -\frac{1}{2}\kappa \left[\hat{\mathscr{C}}_{2SU(3)} - \frac{3}{4}\hat{\mathscr{C}}_{2SO(3)} \right] . \tag{42}$$

In this case, the eigenstates can be labeled by (λ, μ) which characterize the irreducible representations of $SU(3)$. The eigenvalues are

$$E_2 = -\frac{1}{2}\kappa \left[\lambda(\lambda+3) + \mu(\mu+3) + \lambda\mu - \frac{3}{4}L(L+1) \right] . \tag{43}$$

The energy spectrum is characterized by a series of bands, in which the energy spacing is proportional to $L(L+1)$, as in the rigid rotor model. The ground state band has $(\lambda, \mu) = (2N, 0)$ and the first excited band $(2N-4, 2)$ corresponds to a degenerate β and γ band (see Fig. 5). The sign of the coefficient χ is related to a prolate (-) or an oblate (+) deformation.

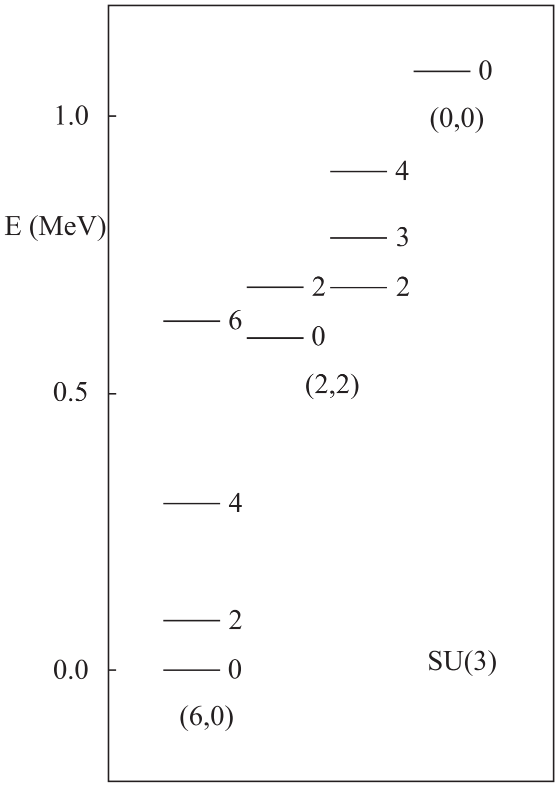

FIGURE 5. Schematic energy spectrum of an even-even nucleus with $SU(3)$ symmetry and $N = 3$. The quantum numbers (λ, μ) are shown below each band and the angular momentum L of each state on the right.

The SO(6) limit

For $\chi = 0$, the Hamiltonian has a $SO(6)$ dynamical symmetry

$$H_3 \;=\; -\kappa\,\hat{Q}(0)\cdot\hat{Q}(0) \;=\; -\kappa\left[\hat{\mathscr{C}}_{2SO(6)} - \hat{\mathscr{C}}_{2SO(5)}\right] \,. \tag{44}$$

The basis states are labeled by σ and τ which characterize the irreducible representations of $SO(6)$ and $SO(5)$, respectively. Characteristic features of the energy spectrum

$$E_3 \;=\; -\kappa\left[\sigma(\sigma+4) - \tau(\tau+3)\right] \,, \tag{45}$$

are the repeating patterns $L = 0,2,4,2$ which is typical of the γ-unstable rotor (see Fig. 6).

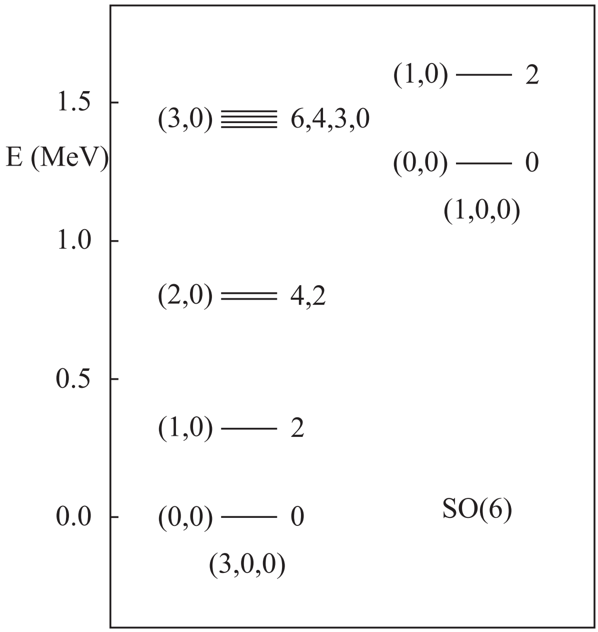

FIGURE 6. Schematic energy spectrum of an even-even nucleus with $SO(6)$ symmetry and $N = 3$. The quantum numbers $(\sigma_1,\sigma_2,\sigma_3) = (\sigma,0,0)$ are shown below each band, the boson seniority $(\tau_1,\tau_2) = (\tau,0)$ is shown on the left and the angular momentum L belonging to each τ multiplet on the right.

For other choices of the coefficients, the Hamiltonian of Eq. (37) describes situations in between any of the dynamical symmetries which correspond to transitional regions, e.g. the Pt-Os isotopes exhibit a transition between a γ-unstable and a rigid rotor $SO(6) \leftrightarrow SU(3)$, the Sm isotopes between vibrational and rotational nuclei $U(5) \leftrightarrow SU(3)$, and the Ru isotopes between vibrational and γ-unstable nuclei $U(5) \leftrightarrow SO(6)$ [21].

Dynamical symmetries in odd-even nuclei

For odd-mass nuclei the IBM has been extended to include single-particle degrees of freedom [22]. The Interacting Boson-Fermion Model (IBFM) has as its building blocks a set of N bosons with $l = 0,2$ and an odd nucleon $M = 1$ (either a proton or a neutron) occupuying the single-particle orbits with angular momenta $j = j_1, j_2, \ldots$. The components of the fermion angular momenta span the Ω-dimensional space of the group $U(\Omega)$ with $\Omega = \sum_j (2j+1)$.

One introduces, in addition to the boson creation $b_i^\dagger$ and annihilation b_i operators for the collective degrees of freedom, fermion creation $a_\mu^\dagger$ and annihilation a_μ operators for the single-particle. The fermion operators satisfy anti-commutation relations

$$\{a_\mu, a_\nu^\dagger\} \;=\; \delta_{\mu\nu} \;, \qquad \{a_\mu^\dagger, a_\nu^\dagger\} \;=\; \{a_\mu, a_\nu\} \;=\; 0 \;. \tag{46}$$

By construction, the fermion operators commute with the boson operators. The bilinear products

$$A_{\mu\nu} \;=\; a_\mu^\dagger a_\nu \;, \tag{47}$$

generate the algebra of $U(\Omega)$, the unitary group in Ω dimensions

$$[A_{\mu\nu}, A_{\rho\sigma}] \;=\; A_{\mu\sigma}\,\delta_{\nu\rho} - A_{\rho\nu}\,\delta_{\mu\sigma} \;. \tag{48}$$

For the mixed system of boson and fermion degrees of freedom we introduce angular-momentum-coupled generators as

$$\begin{aligned} B_\mu^{(\lambda)}(l,l') &= (b_l^\dagger \tilde{b}_{l'})_\mu^{(\lambda)} \;, \\ A_\mu^{(\lambda)}(j,j') &= (a_j^\dagger \tilde{a}_{j'})_\mu^{(\lambda)} \;, \end{aligned} \tag{49}$$

where $\tilde{a}_{jm}$ is defined to be a spherical tensor operator

$$\tilde{a}_{jm} \;=\; (-)^{j-m} a_{j,-m} \;. \tag{50}$$

The most general one- and two-body rotational invariant Hamiltonian of the IBFM can be written as

$$H \;=\; H_B + H_F + V_{BF} \;, \tag{51}$$

where H_B is the IBM Hamiltonian of Eq. (35), H_F is the fermion Hamiltonian

$$H_F \;=\; \sum_j \eta_j \sum_m a_{jm}^\dagger a_{jm} + \sum_\lambda \sum_{j_1 j_2 j_3 j_4} v_{j_1 j_2 j_3 j_4}^{(\lambda)} \left[(a_{j_1}^\dagger \tilde{a}_{j_2})^{(\lambda)} \cdot (a_{j_3}^\dagger \tilde{a}_{j_4})^{(\lambda)} + h.c. \right] \;, \tag{52}$$

and V_{BF} the boson-fermion interaction

$$V_{BF} \;=\; \sum_\lambda \sum_{l_1 l_2 j_1 j_2} w_{l_1 l_2 j_1 j_2}^{(\lambda)} \left[(b_{l_1}^\dagger \tilde{b}_{l_2})^{(\lambda)} \cdot (a_{j_1}^\dagger \tilde{a}_{j_2})^{(\lambda)} + h.c. \right] \;. \tag{53}$$

The IBFM Hamiltonian has an interesting algebraic structure, that suggests the possible occurrence of dynamical symmetries in odd-A nuclei. Since in the IBFM odd-A

nuclei are described in terms of a mixed system of interacting bosons and fermions, the concept of dynamical symmetries has to be generalized. Under the restriction, that both the boson and fermion states have good angular momentum, the respective group chains should contain the rotation group ($SO(3)$ for bosons and $SU(2)$ for fermions) as a subgroup

$$\begin{array}{lll} U^B(6) & \supset \cdots \supset & SO^B(3) \\ U^F(\Omega) & \supset \cdots \supset & SU^F(2) \end{array} \tag{54}$$

where we have introduced superscripts to distinguish between boson and fermion groups. If one of subgroups of $U^B(6)$ is isomorphic to one of the subgroups of $U^F(\Omega)$, the boson and fermion group chains can be combined into a common boson-fermion group chain. When the Hamiltonian is written in terms of Casimir invariants of the combined boson-fermion group chain, a dynamical boson-fermion symmetry arises.

The $Spin(6)$ limit

Among the many different possibilities, we consider two dynamical boson-fermion symmetries associated with the $SO(6)$ limit of the IBM. The first example discussed in the literature [23, 32] is the case of bosons with $SO(6)$ symmetry and the odd nucleon occupying a single-particle orbit with spin $j = 3/2$. The relevant group chains are

$$\begin{array}{lllllll} U^B(6) & \supset & SO^B(6) & \supset & SO^B(5) & \supset & SO^B(3) \\ U^F(4) & \supset & SU^F(4) & \supset & Sp^F(4) & \supset & SU^F(2) \end{array} \tag{55}$$

Since $SO(6)$ and $SU(4)$ are isomorphic, the boson and fermion group chains can be combined into

$$\begin{array}{lll} U^B(6) \otimes U^F(4) & \supset & SO^B(6) \otimes SU^F(4) \\ & \supset & Spin(6) \supset Spin(5) \supset Spin(3) \ . \end{array} \tag{56}$$

The spinor groups $Spin(n)$ are the universal covering groups of the orthogonal groups $SO(n)$, with $Spin(6) \sim SU(4)$, $Spin(5) \sim Sp(4)$ and $Spin(3) \sim SU(2)$. The generators of the spinor groups consist of the sum of a boson and a fermion part. For example, for the quadrupole operator we have

$$\hat{Q} \;=\; (s^\dagger \tilde{d} + d^\dagger \tilde{s})^{(2)} + (a^\dagger_{3/2} \tilde{a}_{3/2})^{(2)} \ . \tag{57}$$

We consider a simple quadrupole-quadrupole interaction which, just as for the $SO(6)$ limit of the IBM, can be written as the difference of two Casimir invariants

$$H_1 \;=\; -\kappa \, \hat{Q} \cdot \hat{Q} \;=\; -\kappa \left[\hat{\mathscr{C}}_{2Spin(6)} - \hat{\mathscr{C}}_{2Spin(5)} \right] \ . \tag{58}$$

The basis states are classified by $(\sigma_1, \sigma_2, \sigma_3)$, (τ_1, τ_2) and J which label the irreducible representations of the spinor groups $Spin(6)$, $Spin(5)$ and $Spin(3)$. The energy spectrum is obtained from the expectation value of the Casimir invariants of the spinor groups

$$E_1 \;=\; -\kappa \left[\sigma_1(\sigma_1+4) + \sigma_2(\sigma_2+2) + \sigma_3^2 - \tau_1(\tau_1+3) - \tau_2(\tau_2+1) \right] \ . \tag{59}$$

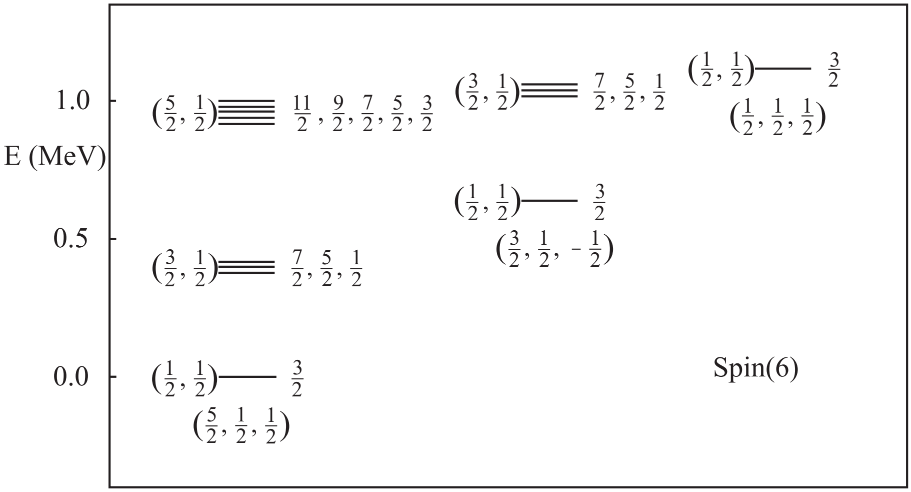

FIGURE 7. Schematic energy spectrum of an odd-even nucleus with $Spin(6)$ symmetry with $N = 2$ and $M = 1$. The quantum numbers $(\sigma_1, \sigma_2, \sigma_3)$ are shown below each band, the labels (τ_1, τ_2) are shown on the left and the angular momentum J belonging to each (τ_1, τ_2) multiplet on the right.

The energy spectrum is characterized by a series of bands labeled by $(\sigma_1, \sigma_2, \sigma_3)$, whose rotational energies depend on the values of (τ_1, τ_2) (see Fig. 7).

The mass region of the Os-Ir-Pt-Au nuclei, where the even-even Pt nuclei are well described by the $SO(6)$ limit of the IBM and the odd proton mainly occupies the $2d_{3/2}$ shell, provide experimental examples of this symmetry, *e.g.* ^{191,193}Ir and ^{193,195}Au [23, 32]. Fig. 8 shows the spectrum of the positive parity levels of the nucleus ^{191}Ir as an example of the $Spin(6)$ limit.

The $U(6) \otimes SU(2)$ *limit*

The concept of dynamical boson-fermion symmetries is not restricted to cases in which the odd nucleon occupies a single-j orbit. The first example of a multi-j case discussed in the literature [25] is that of a dynamical boson-fermion symmetry associated with the $SO(6)$ limit and the odd nucleon occupying single-particle orbits with spin $j = 1/2$, 3/2, 5/2. In this case, the fermion space is decomposed into a pseudo-orbital part with $k = 0, 2$ and a pseudo-spin part with $s = 1/2$ corresponding to the group reduction

$$U^F(12) \supset U^F(6) \otimes U^F(2) \supset \begin{cases} U^F(5) \otimes U^F(2) \\ SU^F(3) \otimes U^F(2) \\ SO^F(6) \otimes U^F(2) \end{cases} \qquad (60)$$

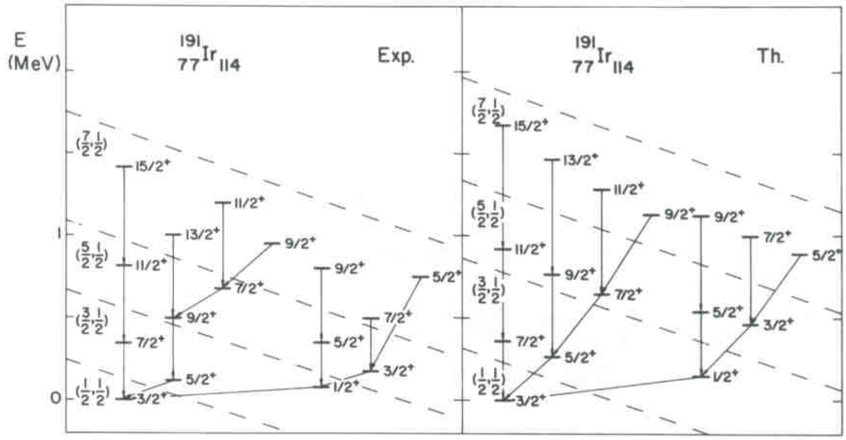

FIGURE 8. Example of an odd-even nucleus with $Spin(6)$ symmetry [32].

Since the pseudo-orbital angular momentum k has the same values as the angular momentum of the s- and d- bosons of the IBM, it is clear that the pseudo-orbital part can be combined with all three dynamical symmetries of the IBM

$$U^B(6) \supset \begin{cases} U^B(5) \\ SU^B(3) \\ SO^B(6) \end{cases} . \tag{61}$$

into a dynamical boson-fermion symmetry. The case, in which the bosons have $SO(6)$ symmetry is of particular interest, since the negative parity states in Pt with the odd neutron occupying the $3p_{1/2}$, $3p_{3/2}$ and $3f_{5/2}$ orbits have been suggested as possible experimental examples of a multi-j boson-fermion symmetry. In this case, the relevant boson-fermion group chain is

$$\begin{aligned} U^B(6) \otimes U^F(12) &\supset U^B(6) \otimes U^F(6) \otimes U^F(2) \\ &\supset U^{BF}(6) \otimes U^F(2) \\ &\supset SO^{BF}(6) \otimes U^F(2) \\ &\supset SO^{BF}(5) \otimes U^F(2) \\ &\supset SO^{BF}(3) \otimes SU^F(2) \\ &\supset Spin(3) . \end{aligned} \tag{62}$$

Just as in the first example for the spinor groups, the generators of the boson-fermion groups consist of the sum of a boson and a fermion part, *e.g.* the quadrupole operator is now written as

$$\hat{Q} = (s^\dagger \tilde{d} + d^\dagger \tilde{s})^{(2)} + \sqrt{\frac{4}{5}}\,(a^\dagger_{3/2}\tilde{a}_{1/2} - a^\dagger_{1/2}\tilde{a}_{3/2})^{(2)} + \sqrt{\frac{6}{5}}\,(a^\dagger_{5/2}\tilde{a}_{1/2} + a^\dagger_{1/2}\tilde{a}_{5/2})^{(2)} \ . \tag{63}$$

Also in this case, the quadrupole-quadrupole interaction can be written as the difference of two Casimir invariants

$$H_2 = -\kappa \hat{Q}\cdot\hat{Q} = -\kappa \left[\hat{\mathscr{C}}_{2SO^{BF}(6)} - \hat{\mathscr{C}}_{2SO^{BF}(5)} \right] \ . \tag{64}$$

The basis states are classified by $[N_1,N_2]$, $(\sigma_1,\sigma_2,\sigma_3)$, (τ_1,τ_2) and L which label the irreducible representations of the boson-fermion groups $U^{BF}(6)$, $SO^{BF}(6)$, $SO^{BF}(5)$ and $SO^{BF}(3)$, respectively. The total angular momentum is given by $\vec{J} = \vec{L} + \vec{s}$. The corresponding energy formula has the same form as for the previous case

$$E_2 = -\kappa \left[\sigma_1(\sigma_1+4) + \sigma_2(\sigma_2+2) + \sigma_3^2 - \tau_1(\tau_1+3) - \tau_2(\tau_2+1) \right] \ . \tag{65}$$

However, the allowed values of the quantum numbers are different. Fig. 9 shows a typical spectrum in the $U(6)\otimes U(2)$ limit. The spectrum consists of a series of bands labeled by $[N_1,N_2]$, $(\sigma_1,\sigma_2,\sigma_3)$.

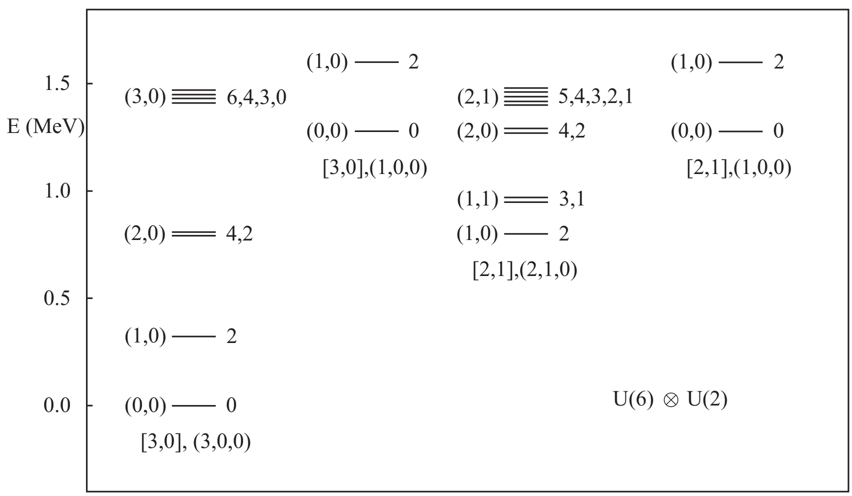

FIGURE 9. Schematic energy spectrum of an odd-even nucleus with $U(6)\otimes U(2)$ symmetry for $N=2$ and $M=1$. The quantum numbers $[N_1,N_2]$, $(\sigma_1,\sigma_2,\sigma_3)$ are shown below each band, the labels (τ_1,τ_2) are shown on the left and the angular momentum L belonging to each (τ_1,τ_2) multiplet on the right. All levels are doublets with $J = L \pm \frac{1}{2}$ with the exception of $L=0$ for which $J=\frac{1}{2}$ only.

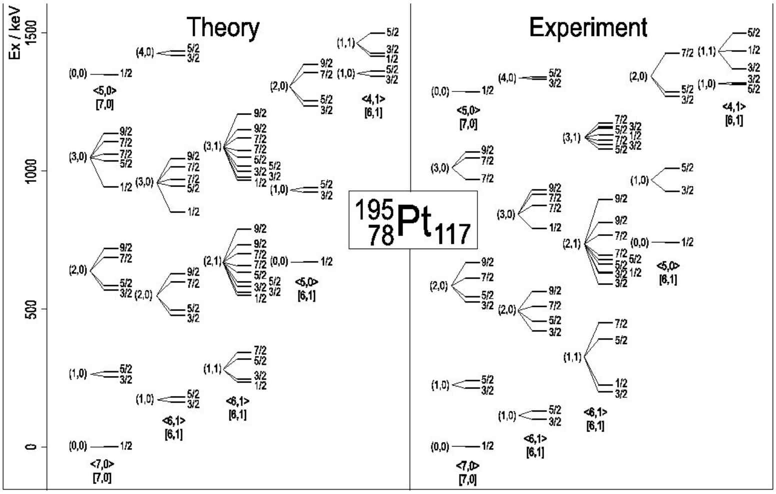

FIGURE 10. Example of an odd-even nucleus with $U(6) \otimes U(2)$ symmetry.

The mass region of the Os-Ir-Pt-Au nuclei, where the even-even Pt nuclei are well described by the $SO(6)$ limit of the IBM and the odd neutron mainly occupies the negative parity orbits $3p_{1/2}$, $3p_{3/2}$ and $3f_{5/2}$ provides experimental examples of this symmetry, in particular the negative parity levels of ^{195}Pt are very well described by the $U(6) \otimes U(2)$ limit of the IBFM [25, 28, 33, 34].

Dynamical symmetries in odd-odd nuclei

For odd-odd nuclei the IBM has to be extended to include the single-particle degrees of freedom of both an odd proton and an odd neutron. The ensuing Interacting Boson-Fermion-Fermion Model (IBFFM) has as its building blocks a set of N bosons with $l = 0, 2$, an odd proton and an odd neutron, both of which can occupy a certain number of single-particle orbits. The components of the fermion angular momenta span the $\Omega_\nu \Omega_\pi$-dimensional space of the group $U(\Omega_\nu) \otimes U(\Omega_\pi)$ with $\Omega_\nu = \sum_{j_\nu}(2j_\nu + 1)$ and and $\Omega_\pi = \sum_{j_\pi}(2j_\pi + 1)$.

The most general one- and two-body rotational invariant Hamiltonian of the IBFFM can be written as

$$H \;=\; H_B + H_{F_\nu} + H_{F_\pi} + V_{F_\nu F_\pi} + V_{BF_\nu} + V_{BF_\pi} \;, \tag{66}$$

where H_B is the IBM Hamiltonian of Eq. (35), and H_{F_ν} and H_{F_π} denote the fermion Hamiltonian of Eq. (52) for the odd neutron and proton, respectively. $V_{F_\nu F_\pi}$ represents the interaction between the odd proton and the odd neutron

$$V_{F_\nu F_\pi} \;=\; \sum_\lambda \sum_{j_1 j_2 j_3 j_4} x^{(\lambda)}_{j_1 j_2 j_3 j_4} \left[(a^\dagger_{j_1} \tilde{a}_{j_2})^{(\lambda)}_\nu \cdot (a^\dagger_{j_3} \tilde{a}_{j_4})^{(\lambda)}_\pi + h.c. \right] \;. \tag{67}$$

Finally, V_{BF_ν} and V_{BF_π} denote the boson-fermion interaction of Eq. (53) for the interaction between the even-even core and the odd neutron and proton, respectively.

Also the IBFFM Hamiltonian has dynamical symmetries in which the eigenvalue problem can be solved in closed analytic form. Here we study a special case that is a combination of the $SO(6)$ limit in even-even nuclei, the $Spin(6)$ limit in odd-proton nuclei, and the $U(6)\otimes U(2)$ limit in odd-neutron nuclei. The relevant group chains are

$$\begin{array}{ccccccccccc} & & U^B(6) & & & \supset & SO^B(6) & & \\ U^{F_\nu}(12) & \supset & U^{F_\nu}(6) & \otimes & U^{F_\nu}(2) & \supset & SO^{F_\nu}(6) & \otimes & U^{F_\nu}(2) \\ U^{F_\pi}(4) & \supset & & & & & SU^{F_\pi}(4) & & \end{array} \tag{68}$$

There are many different ways to couple the three group chains, but the coupling scheme that is most relevant in describing the spectra of complex nuclei is the one in which first the odd neutron is coupled to the boson core at the level of $U(6)$ as in Eq. (62), and next the odd-proton is coupled at the level of $SO(6)\sim SU(4)\sim Spin(6)$ as in Eq. (56) to obtain the following group chain

$$\begin{aligned} U^B(6) \;\otimes\; & U^{F_\nu}(12)\otimes U^{F_\pi}(4) \\ \supset\; & U^B(6)\otimes U^{F_\nu}(6)\otimes U^{F_\nu}(2)\otimes U^{F_\pi}(4) \\ \supset\; & U^{BF_\nu}(6)\otimes U^{F_\nu}(2)\otimes U^{F_\pi}(4) \\ \supset\; & SO^{BF_\nu}(6)\otimes U^{F_\nu}(2)\otimes SU^{F_\pi}(4) \\ \supset\; & Spin(6)\otimes U^{F_\nu}(2) \\ \supset\; & Spin(5)\otimes U^{F_\nu}(2) \\ \supset\; & Spin(3)\otimes SU^{F_\nu}(2) \\ \supset\; & SU(2)\;. \end{aligned} \tag{69}$$

Again, let's consider a quadrupole-quadrupole interaction. The quadrupole operator is now the sum of a collective part and a single-particle part for the odd proton and the odd neutron

$$\begin{aligned} \hat{Q} \;=\; & (s^\dagger\tilde{d}+d^\dagger\tilde{s})^{(2)}+(a^\dagger_{3/2}\tilde{a}_{3/2})^{(2)}_\pi \\ & +\sqrt{\frac{4}{5}}\,(a^\dagger_{3/2}\tilde{a}_{1/2}-a^\dagger_{1/2}\tilde{a}_{3/2})^{(2)}_\nu+\sqrt{\frac{6}{5}}\,(a^\dagger_{5/2}\tilde{a}_{1/2}+a^\dagger_{1/2}\tilde{a}_{5/2})^{(2)}_\nu\;. \end{aligned} \tag{70}$$

Just as in the previous examples the quadrupole-quadrupole interaction can be written as the difference of two Casimir invariants

$$H \;=\; -\kappa\,\hat{Q}\cdot\hat{Q} \;=\; -\kappa\left[\hat{\mathscr{C}}_{2Spin(6)}-\hat{\mathscr{C}}_{2Spin(5)}\right]\;, \tag{71}$$

and the corresponding energy eigenvalues are again given by

$$E \;=\; -\kappa\left[\sigma_1(\sigma_1+4)+\sigma_2(\sigma_2+2)+\sigma_3^2-\tau_1(\tau_1+3)-\tau_2(\tau_2+1)\right]\;. \tag{72}$$

The basis states are classified by $[N_1,N_2]$, $(\Sigma_1,\Sigma_2,\Sigma_3)$, $(\sigma_1,\sigma_2,\sigma_3)$, (τ_1,τ_2), J and L which label the irreducible representations of the boson-fermion groups $U^{BF_\nu}(6)$,

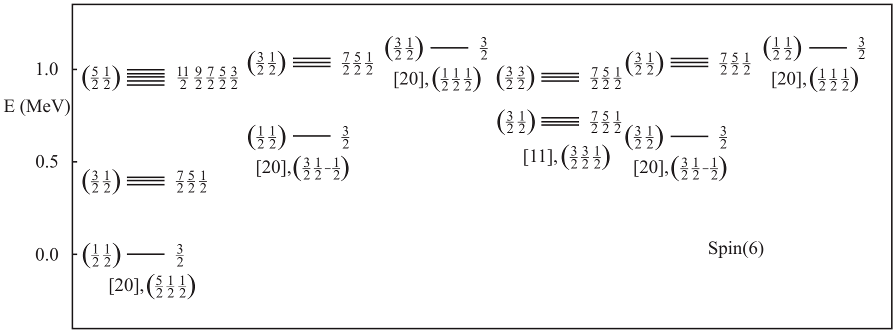

FIGURE 11. Schematic energy spectrum of an odd-odd nucleus with $Spin(6)$ symmetry for $N = 1$ and $M_\nu = M_\pi = 1$. The quantum numbers $[N_1, N_2]$, $(\sigma_1, \sigma_2, \sigma_3)$ are shown below each band, the labels (τ_1, τ_2) are shown on the left and the angular momentum J belonging to each (τ_1, τ_2) multiplet on the right. All levels are doublets with $L = J \pm \frac{1}{2}$.

$SO^{BF_\nu}(6)$, $Spin(6)$, $Spin(5)$, $Spin(3)$ and $SU(2)$, respectively. In this case, the total angular momentum is denoted by $\vec{L} = \vec{J} + \vec{s}$ (L is integer and J half-integer). The spectrum is characterized by a sequence of bands labeled by $[N_1, N_2]$, $(\sigma_1, \sigma_2, \sigma_3)$ (see Fig. 11).

The mass region of the Os-Ir-Pt-Au nuclei, where the even-even Pt nuclei are well described by the $SO(6)$ limit of the IBM and the odd neutron mainly occupies the negative parity orbits $3p_{1/2}$, $3p_{3/2}$ and $3f_{5/2}$ (see Fig. 10) and the odd proton the positive parity orbit $2d_{3/2}$ (see Fig. 8) provides experimental examples of this symmetry, in particular the odd-odd nuclei ^{196}Au and ^{194}Ir [29, 35]. In Fig. 12 we show the results for ^{196}Au.

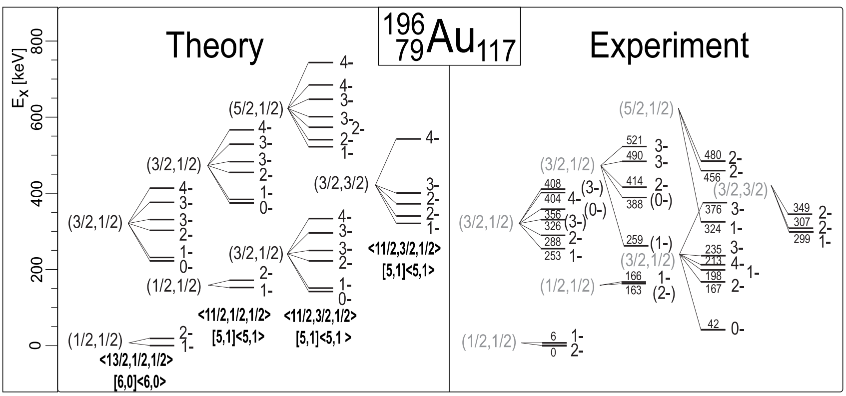

FIGURE 12. Example of an odd-odd nucleus with $Spin(6)$ symmetry [29].

Dynamical supersymmetries

Boson-fermion symmetries can further be extended by introducing the concept of supersymmetries [24], in which states in both even-even and odd-even nuclei are treated in a single framework. In the previous section, we have discussed the symmetry properties of a mixed system of boson and fermion degrees of freedom for a fixed number of bosons N and one fermion $M=1$. The operators B_{ij} and $A_{\mu\nu}$

$$B_{ij}=b_i^\dagger b_j \ , \qquad A_{\mu\nu}=a_\mu^\dagger a_\nu \ , \tag{73}$$

which generate the Lie algebra of the symmetry group $U^B(6)\otimes U^F(\Omega)$ of the IBFM, can only change bosons into bosons and fermions into fermions. The number of bosons N and the number of fermions M are both conserved quantities. As explained in Section 2.6, in addition to B_{ij} and $A_{\mu\nu}$, one can introduce operators that change a boson into a fermion and *vice versa*

$$F_{i\mu}=b_i^\dagger a_\mu \ , \qquad G_{\mu i}=a_\mu^\dagger b_i \ . \tag{74}$$

The enlarged set of operators B_{ij}, $A_{\mu\nu}$, $F_{i\mu}$ and $G_{\mu i}$ forms a closed (super)algebra which consists of both commutation and anticommutation relations

$$\begin{aligned}
\left[B_{ij},B_{kl}\right] &= B_{il}\delta_{jk}-B_{kj}\delta_{il} \ , \\
\left[A_{\mu\nu},A_{\rho\sigma}\right] &= A_{\mu\sigma}\delta_{\nu\rho}-A_{\rho\nu}\delta_{\mu\sigma} \ , \\
\left[B_{ij},A_{\mu\nu}\right] &= 0 \ , \\
\left[B_{ij},F_{k\mu}\right] &= F_{i\mu}\delta_{jk} \ , \\
\left[G_{\mu i},B_{kl}\right] &= G_{\mu l}\delta_{ik} \ , \\
\left[F_{i\mu},A_{\rho\sigma}\right] &= F_{i\sigma}\delta_{\mu\rho} \ , \\
\left[A_{\mu\nu},G_{\rho i}\right] &= G_{\mu i}\delta_{\nu\rho} \ , \\
\left\{F_{i\mu},G_{\nu j}\right\} &= B_{ij}\delta_{\mu\nu}+A_{\nu\mu}\delta_{ij} \ , \\
\left\{F_{i\mu},F_{j\nu}\right\} &= 0 \ , \\
\left\{G_{\mu i},G_{\nu j}\right\} &= 0 \ .
\end{aligned} \tag{75}$$

This algebra can be identified with that of the graded Lie group $U(6/\Omega)$. It provides an elegant scheme in which the IBM and IBFM can be unified into a single framework [24]

$$U(6/\Omega)\supset U^B(6)\otimes U^F(\Omega) \ . \tag{76}$$

In this supersymmetric framework, even-even and odd-mass nuclei form the members of a supermultiplet which is characterized by $\mathcal{N}=N+M$, i.e. the total number of bosons and fermions. Supersymmetry thus distinguishes itself from "normal" symmetries in that it includes, in addition to transformations among fermions and among bosons, also transformations that change a boson into a fermion and *vice versa* (see Table 1).

TABLE 1. Overview of algebraic models.

Model	Generators	Invariant	Symmetry
IBM	$b_i^\dagger b_j$	N	$U(6)$
IBFM	$b_i^\dagger b_j$, $a_\mu^\dagger a_\nu$	N, M	$U(6) \otimes U(\Omega)$
n-SUSY	$b_i^\dagger b_j$, $a_\mu^\dagger a_\nu$, $b_i^\dagger a_\mu$, $a_\mu^\dagger b_i$	$\mathscr{N}$	$U(6/\Omega)$

$U(6/4)$ *supersymmetry*

The Os-Ir-Pt-Au mass region provides ample experimental evidence for the occurrence of dynamical (super)symmetries in nuclei. The even-even nuclei ^{194,196}Pt are the standard examples of the $SO(6)$ limit of the IBM [36] and the odd proton, in first approximation, occupies the single-particle level $2d_{3/2}$. In this special case, the boson and fermion groups can be combined into spinor groups, and the odd-proton nuclei ^{191,193}Ir and ^{193,195}Au were suggested as examples of the $Spin(6)$ limit [23, 32]. The appropriate extension to a supersymmetry is by means of the graded Lie group $U(6/4)$

$$\begin{aligned} U(6/4) &\supset U^B(6) \otimes U^F(4) \supset SO^B(6) \otimes SU^F(4) \\ &\supset Spin(6) \supset Spin(5) \supset Spin(3) \; . \end{aligned} \tag{77}$$

A dynamical supersymmetry arises when the Hamiltonian is expressed in terms of the Casimir invariants of the subgroups of $U(6/4)$

$$H_1 \;=\; -A\,\hat{\mathscr{C}}_{2Spin(6)} + B\,\hat{\mathscr{C}}_{2Spin(5)} + C\,\hat{\mathscr{C}}_{2Spin(3)} \; . \tag{78}$$

The energy spectrum is given by the expectation value of the Casimir invariants of the spinor groups

$$\begin{aligned} E_1 \;=\; & -A\left[\sigma_1(\sigma_1+4)+\sigma_2(\sigma_2+2)+\sigma_3^2\right] \\ & +B\left[\tau_1(\tau_1+3)+\tau_2(\tau_2+1)\right]+C\,J(J+1) \; , \end{aligned} \tag{79}$$

which simultaneously describes the spectra of both the even-even and the odd-even nucleus with a single set of parameters A, B and C.

The pairs of nuclei ^{190}Os - ^{191}Ir, ^{192}Os - ^{193}Ir, ^{192}Pt - ^{193}Au and ^{194}Pt - ^{195}Au were analyzed as examples of a $U(6/4)$ supersymmetry [24]. In Fig. 13, we show the results for the pair ^{190}Os - ^{191}Ir.

$U(6/12)$ *supersymmetry*

Another example of a dynamical supersymmetry in this mass region is that of the Pt nuclei. The even-even isotopes are well described by the $SO(6)$ limit of the IBM and the

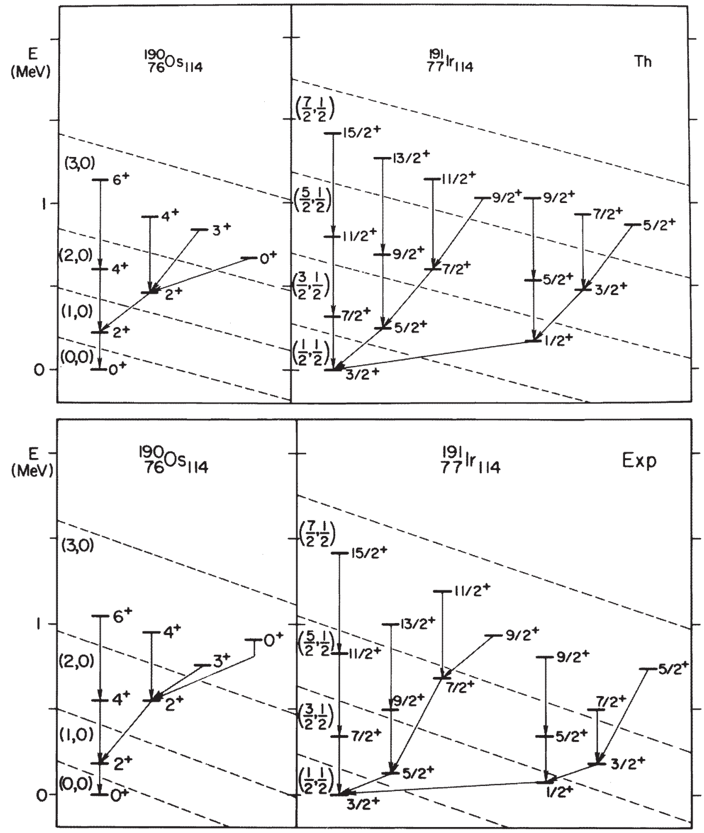

FIGURE 13. Example of pair of nuclei with $U(6/4)$ supersymmetry [24].

odd neutron mainly occupies the negative parity orbits $3p_{1/2}$, $3p_{3/2}$ and $3f_{5/2}$. In this case, the graded Lie group is $U(6/12)$

$$
\begin{aligned}
U(6/12) &\supset U^B(6) \otimes U^F(12) \\
&\supset U^B(6) \otimes U^F(6) \otimes U^F(2) \\
&\supset U^{BF}(6) \otimes U^F(2) \\
&\supset SO^{BF}(6) \otimes U^F(2) \\
&\supset SO^{BF}(5) \otimes U^F(2) \\
&\supset SO^{BF}(3) \otimes SU^F(2) \\
&\supset Spin(3) \, . \qquad (80)
\end{aligned}
$$

In this case, the Hamiltonian

$$H_2 \;=\; \alpha \hat{\mathscr{C}}_{2U^{BF}(6)} + \beta \hat{\mathscr{C}}_{2SO^{BF}(6)} + \gamma \hat{\mathscr{C}}_{2SO^{BF}(5)}$$

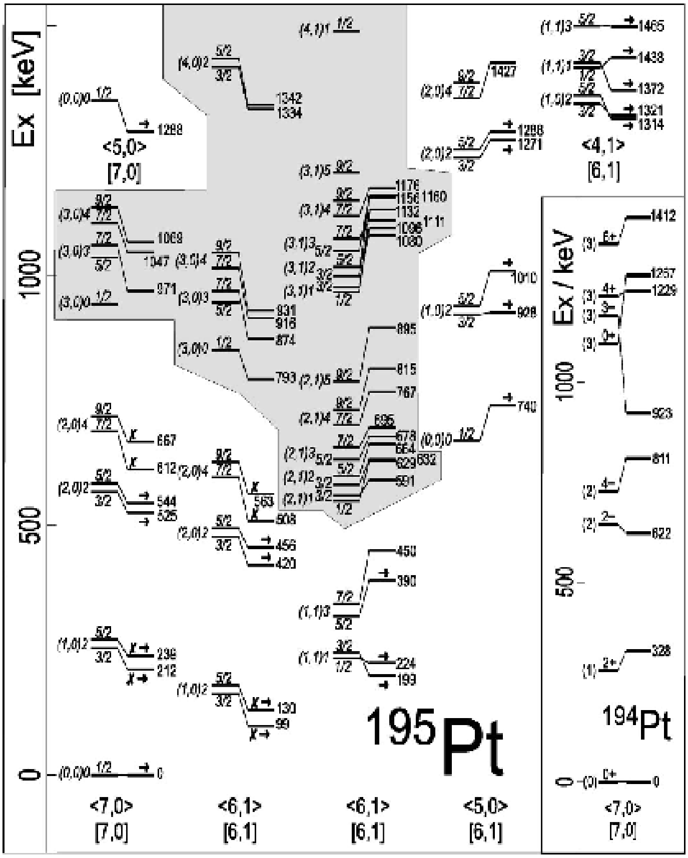

FIGURE 14. Example of a pair of nuclei with $U(6/12)$ supersymmetry [28].

$$+\delta\hat{\mathscr{C}}_{2SO^{BF}(3)}+\varepsilon\hat{\mathscr{C}}_{2Spin(3)}\ , \tag{81}$$

simultaneously describes the excitation spectra of both the even-even and the odd-even nucleus with a single set of parameters α, β, γ, δ and ε. . The energy spectrum is given by the eigenvalues of the Casimir operators

$$\begin{aligned} E_2 \;&=\; \alpha\left[N_1(N_1+5)+N_2(N_2+3)\right]+\beta\left[\sigma_1(\sigma_1+4)+\sigma_2(\sigma_2+2)+\sigma_3^2\right] \\ &\quad +\gamma\left[\tau_1(\tau_1+3)+\tau_2(\tau_2+1)\right]+\delta L(L+1)+\varepsilon J(J+1)\ . \end{aligned} \tag{82}$$

The odd-neutron nucleus ^{195}Pt and the even-even nucleus ^{194}Pt were studied as an example of a $U(6/12)$ supersymmetry (see Fig. 14) [25, 33, 34, 28].

TABLE 2. The number of bosons and fermions in a supersymmetric quartet of nuclei.

Nucleus	N_π	M_π	N_ν	M_ν	Nucleus	N_π	M_π	N_ν	M_ν
${}^{194}_{78}\mathrm{Pt}_{116}$	2	0	5	0	${}^{192}_{76}\mathrm{Os}_{116}$	3	0	5	0
${}^{195}_{78}\mathrm{Pt}_{117}$	2	0	4	1	${}^{193}_{76}\mathrm{Os}_{117}$	3	0	4	1
${}^{195}_{79}\mathrm{Au}_{116}$	1	1	5	0	${}^{193}_{77}\mathrm{Ir}_{116}$	2	1	5	0
${}^{196}_{79}\mathrm{Au}_{117}$	1	1	4	1	${}^{194}_{77}\mathrm{Ir}_{117}$	2	1	4	1

Dynamical neutron-proton supersymmetries

As we have seen in the previous section, the mass region $A \sim 190$ has been a rich source of possible empirical evidence for the existence of (super)symmetries in nuclei. The pairs of nuclei ^{190}Os - ^{191}Ir, ^{192}Os - ^{193}Ir, ^{192}Pt - ^{193}Au and ^{194}Pt - ^{195}Au have been analyzed as examples of a $U(6/4)$ supersymmetry [24], and the nuclei ^{194}Pt - ^{195}Pt as an example of a $U(6/12)$ supersymmetry [25]. These ideas were later extended to the case where neutron and proton bosons are distinguished [30], predicting in this way a correlation among quartets of nuclei, consisting of an even-even, an odd-proton, an odd-neutron and an odd-odd nucleus. The best experimental example of such a quartet with $U(6/12)_\nu \otimes U(6/4)_\pi$ supersymmetry is provided by the nuclei ^{194}Pt, ^{195}Au, ^{195}Pt and ^{196}Au.

The supersymmetric classification of nuclear levels in the Pt and Au isotopes has been re-examined by taking advantage of the significant improvements in experimental capabilities developed in the last decade. High resolution transfer experiments with protons and polarized deuterons have strengthened the evidence for the existence of supersymmetry in atomic nuclei. The experiments include high resolution transfer experiments to ^{196}Au at TU/LMU München [27, 28], and in-beam gamma ray and conversion electron spectroscopy following the reactions ${}^{196}\mathrm{Pt}(d,2n)$ and ${}^{196}\mathrm{Pt}(p,n)$ at the cyclotrons of the PSI and Bonn [29]. These studies have achieved an improved classification of states in ^{195}Pt and ^{196}Au which give further support to the original ideas [23, 25, 30] and extend and refine previous experimental work in this research area.

The number of bosons and fermions are related to the number of valence nucleons, *i.e.* the number of protons and neutrons outside the closed shells. The relevant closed shells are $Z = 82$ for protons and $N = 126$ for neutrons. For the even-even nucleus ${}^{194}_{78}\mathrm{Pt}_{116}$ the number of bosons are $N_\pi = (82-78)/2 = 2$ and $N_\nu = (126-116)/2 = 5$. There are no unpaired nucleons $M_\pi = M_\nu = 0$. For the odd-neutron nucleus ${}^{195}_{78}\mathrm{Pt}_{117}$ there are 9 valence neutrons which leads to $N_\nu = 4$ neutron bosons and $M_\nu = 1$ unpaired neutron. The ${}_{79}$Au isotopes have 3 valence protons which are divided over $N_\pi = 1$ proton boson and $M_\pi = 1$ unpaired proton. This supersymmetric quartet of nuclei is characterized by $\mathcal{N}_\pi = N_\pi + M_\pi = 2$ and $\mathcal{N}_\nu = N_\nu + M_\nu = 5$. The number of bosons and fermions are summarized in Table 2.

The relevant subgroup chain of $U(6/12)_\nu \otimes U(6/4)_\pi$ for the neutron-proton (or extended) supersymmetry is given by [30]

$$\begin{aligned}
U(6/12)_\nu \otimes U(6/4)_\pi &\supset U^{B_\nu}(6) \otimes U^{F_\nu}(12) \otimes U^{B_\pi}(6) \otimes U^{F_\pi}(4) \\
&\supset U^{B}(6) \otimes U^{F_\nu}(6) \otimes U^{F_\nu}(2) \otimes U^{F_\pi}(4) \\
&\supset U^{BF_\nu}(6) \otimes U^{F_\nu}(2) \otimes U^{F_\pi}(4) \\
&\supset SO^{BF_\nu}(6) \otimes U^{F_\nu}(2) \otimes SU^{F_\pi}(4) \\
&\supset Spin(6) \otimes U^{F_\nu}(2) \\
&\supset Spin(5) \otimes U^{F_\nu}(2) \\
&\supset Spin(3) \otimes SU^{F_\nu}(2) \\
&\supset SU(2) \, .
\end{aligned} \tag{83}$$

In this case, the Hamiltonian

$$\begin{aligned}
H &= \alpha\, \hat{\mathscr{C}}_{2U^{BF_\nu}(6)} + \beta\, \hat{\mathscr{C}}_{2SO^{BF_\nu}(6)} + \gamma\, \hat{\mathscr{C}}_{2Spin(6)} \\
&\quad + \delta\, \hat{\mathscr{C}}_{2Spin(5)} + \varepsilon\, \hat{\mathscr{C}}_{2Spin(3)} + \eta\, \hat{\mathscr{C}}_{2SU(2)} \, ,
\end{aligned} \tag{84}$$

describes simultaneously the excitation spectra of a quartet of nuclei consisting of an even-even, an odd-proton, an odd-neutron and an odd-odd nucleus. The energy spectrum is given by

$$\begin{aligned}
E &= \alpha\, [N_1(N_1+5) + N_2(N_2+3) + N_3(N_3+1)] \\
&\quad + \beta\, \left[\Sigma_1(\Sigma_1+4) + \Sigma_2(\Sigma_2+2) + \Sigma_3^2\right] \\
&\quad + \gamma\, \left[\sigma_1(\sigma_1+4) + \sigma_2(\sigma_2+2) + \sigma_3^2\right] \\
&\quad + \delta\, [\tau_1(\tau_1+3) + \tau_2(\tau_2+1)] \\
&\quad + \varepsilon\, J(J+1) + \eta\, L(L+1) \, .
\end{aligned} \tag{85}$$

Fig. 15 shows the results for the quartet of Pt and Au nuclei. The coefficients α, β, γ, δ, ε and η were determined in a simultaneous fit of the excitation energies of the four nuclei [29].

In dynamical (super)symmetries closed expressions can be derived for energies, as well as selection rules and intensities for electromagnetic transitions and transfer reactions. Recent work in this area concerns a study of one- and two-nucleon transfer reactions. As a consequence of the supersymmetry, explicit correlations were found between the spectroscopic factors of the one-proton reactions between n-SUSY partners $^{194}\mathrm{Pt} \leftrightarrow ^{195}\mathrm{Au}$ and $^{195}\mathrm{Pt} \leftrightarrow ^{196}\mathrm{Au}$ [37] which can be tested experimentally. The spectroscopic stengths of two-nucleon transfer reactions constitute a stringent test for two-nucleon correlations in the nuclear wave functions. A study in the framework of nuclear supersymmetry led to a set of closed analytic expressions for ratios of spectroscopic factors. Since these ratios are parameter independent they provide a direct test of the wave functions. A comparison between the recently measured $^{198}\mathrm{Hg}(\vec{d},\alpha)^{196}\mathrm{Au}$ reaction [38] and the predictions of the nuclear quartet supersymmetry [39] lends further support to the validity of supersymmetry in nuclear physics.

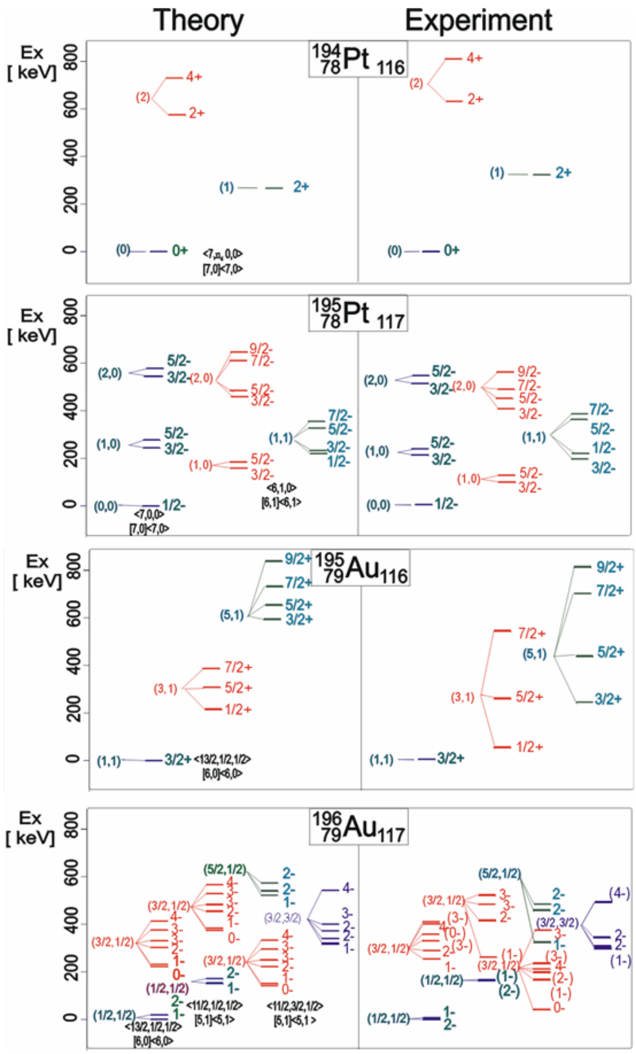

FIGURE 15. Example of a quartet of nuclei with $U(6/12)_\nu \otimes U(6/4)_\pi$ supersymmetry [29].

Two-nucleon transfer reactions

Two-nucleon transfer reactions probe the structure of the final nucleus through the exploration of two-nucleon correlations that may be present. The spectroscopic strengths not only depend on the similarity between the states in the initial and final nucleus, but also on the correlation of the transferred pair of nucleons.

In this section, the recent data on the $^{196}\mathrm{Pt}(\vec{d},\alpha)^{194}\mathrm{Ir}$ reaction [35] are compared with

TABLE 3. Ratios of spectroscopic strengths for $(\vec{d},\alpha)$ reactions $R_{LJ}(\mathrm{ee}\to\mathrm{oo})$ to final states with $(\tau_1,\tau_2)=(\frac{3}{2},\frac{1}{2})$. N is the number of bosons in the odd-odd nucleus ($N=6$ for $^{196}\mathrm{Pt}\to{}^{194}\mathrm{Ir}$).

$[N_1,N_2]$	$(\Sigma_1,\Sigma_2,\Sigma_3)$	$(\sigma_1,\sigma_2,\sigma_3)$	R_{LJ}
$[N,1]$	$(N,1,0)$	$(N+\frac{1}{2},\frac{3}{2},\frac{1}{2})$	1
$[N,1]$	$(N,1,0)$	$(N+\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$\frac{N+4}{15N}$
$[N,1]$	$(N,1,0)$	$(N-\frac{1}{2},\frac{3}{2},-\frac{1}{2})$	$\frac{(N+4)(N+1)(N-1)}{N(N+3)(N+5)}$
$[N,1]$	$(N,1,0)$	$(N-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\frac{(N+1)(N-1)}{15(N+3)(N+5)}$
$[N+1,0]$	$(N+1,0,0)$	$(N+\frac{3}{2},\frac{1}{2},\frac{1}{2})$	$\frac{2(N+4)(N+6)}{15N(N+3)}$
$[N+1,0]$	$(N+1,0,0)$	$(N+\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$\frac{2(N+4)}{15(N+3)}$

the predictions from the $U_\nu(6/12)\otimes U_\pi(6/4)$ supersymmetry. This reaction involves the transfer of a proton-neutron pair, and hence measures the neutron-proton correlation in the odd-odd nucleus. The spectroscopic strengths G_{LJ}

$$G_{LJ}=\left|\sum_{j_\nu j_\pi} g^{LJ}_{j_\nu j_\pi}\left\langle {}^{194}\mathrm{Ir}\left\|(a^\dagger_{j_\nu}a^\dagger_{j_\pi})^{(\lambda)}\right\|{}^{196}\mathrm{Pt}\right\rangle\right|^2 , \tag{86}$$

depend on the reaction mechanism via the coefficients $g^{LJ}_{j_\nu j_\pi}$ and on the nuclear structure part via the reduced matrix elements.

In order to compare with experimental data we calculate the relative strengths $R_{LJ}=G_{LJ}/G^{\mathrm{ref}}_{LJ}$, where G^{ref}_{LJ} is the spectroscopic strength of the reference state. The ratios of spectroscopic strengths to final states with $(\tau_1,\tau_2)=(\frac{3}{2},\frac{1}{2})$ provide a direct test of the nuclear wave functions, since they can only be excited by a single tensor operator [39]. In Table 3 we show the ratios for different final states with $(\tau_1,\tau_2)=(\frac{3}{2},\frac{1}{2})$.

Fig. 16 shows the ratios of spectroscopic strengths of two-nucleon transfer reactions R_{LJ} compared with the theoretical predictions from nuclear supersymmetry. The reference states are easily identified, since they are normalized to one. The calculations were carried out without the introduction of any new parameter since the coefficients $g^{LJ}_{j_\nu j_\pi}$ appearing in the transfer operator of Eq. (86) were taken from the study of the $^{198}\mathrm{Hg}(\vec{d},\alpha)^{196}\mathrm{Au}$ reaction [39]. In general, there is good overall agreement between the experimental and theoretical values, especially if we take into account the simple form of the operator in the calculation of the two-nucleon transfer reaction intensities. The deviations observed in the P_2 and F_2 transfers are most likely due to single-particle configurations outside the model space. For the P_0, P_1, and F_3 distributions the experimental $(\vec{d},\alpha)$ detection limits for weakly populated 0^-, 1^-, and 3^- states prevent a better agreement.

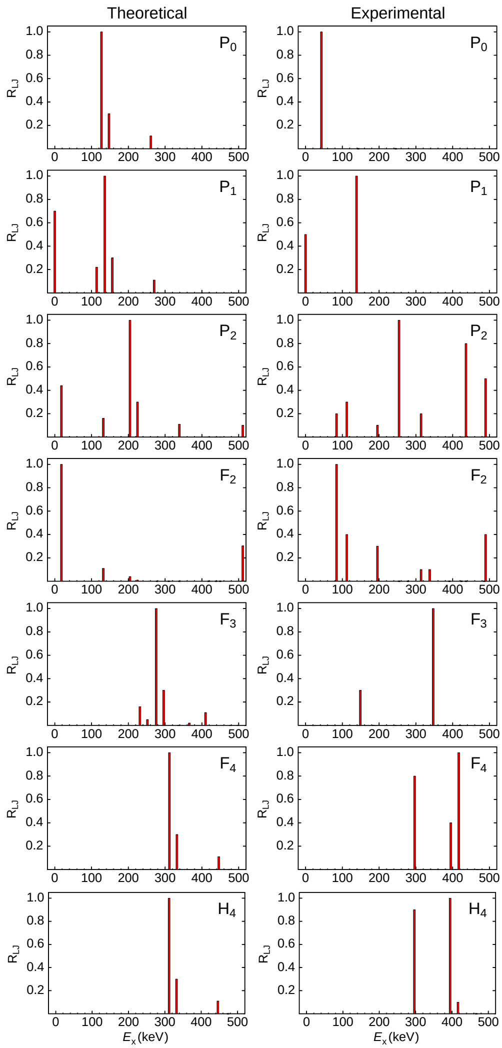

FIGURE 16. Comparison of theoretical (left panels) and experimental values (right panels) of ratios R_{LJ} of spectroscopic strengths.

The new data from the polarized $(\vec{d},\alpha)$ transfer reaction has provided crucial new information about and insight into the structure of the spectrum of ^{194}Ir which led to significant changes in the assignment of levels as compared to previous work [40]. The new assignment agrees with that of the neighboring odd-odd nucleus ^{196}Au [27, 29, 38]. Fig. 17 shows the negative parity levels of ^{194}Ir in comparison with the theoretical spectrum in which it is assumed that these levels originate from the $\nu 3p_{1/2}$, $\nu 3p_{3/2}$, $\nu 2f_{5/2} \otimes \pi 2d_{3/2}$ configuration. Given the complex nature of the spectrum of heavy

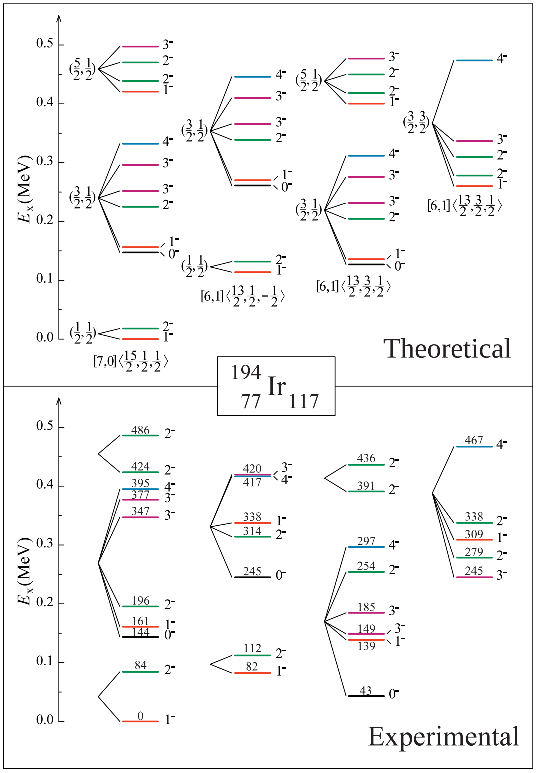

FIGURE 17. Comparison between the theoretical and experimental spectrum of ^{194}Ir [35].

odd-odd nuclei, the agreement is remarkable. There is an almost one-to-one correlation between the experimental and theoretical level schemes [35].

The successful description of the odd-odd nucleus ^{194}Ir opens the possibility of identifying a second quartet of nuclei in the $A \sim 190$ mass region with $U(6/12)_\nu \otimes U(6/4)_\pi$ supersymmetry. The new quartet consists of the nuclei ^{192,193}Os and ^{193,194}Ir and is characterized by $\mathcal{N}_\pi = 3$ and $\mathcal{N}_\nu = 5$ (see Table 2). Whereas the ^{192}Os and ^{193,194}Ir nuclei are well-known experimentally, the available data for ^{193}Os is rather scarce. In Fig. 18 we show the predicted spectrum for ^{193}Os obtained from Eq. (85) using the same parameter set as for ^{194}Ir [35]. The ground state of ^{193}Os has spin and parity $J^P = \frac{3}{2}^-$, which seems to imply that the second band with labels $[7,1]$, $(7,1,0)$ is the ground state band, rather than $[8,0]$, $(8,0,0)$. This ordering of bands is supported by preliminary results from the one-neutron transfer reaction ^{192}Os$(\vec{d},p)^{193}$Os [41].

An analysis of the energy spectra of the four nuclei that make up the quartet shows that the parameter set obtained in 1981 for the pair ^{192}Os-^{193}Ir [24] is very close to that of ^{194}Ir [35], which indicates that the nuclei ^{192,193}Os and ^{193,194}Ir may be interpreted in terms of a quartet of nuclei with $U(6/12)_\nu \otimes U(6/4)_\pi$ supersymmetry.

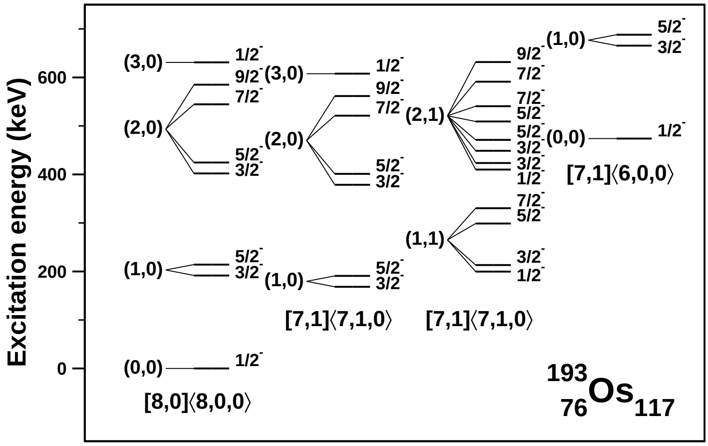

FIGURE 18. Prediction of the spectrum of ^{193}Os for the $U_\nu(6/12) \otimes U_\pi(6/4)$ supersymmetry.

CORRELATIONS

The nuclei belonging to a supersymmetric quartet are described by a single Hamiltonian, and hence the wave functions, transition and transfer rates are strongly correlated. As an example of these correlations, we consider here the transfer reactions between the ^{194,195}Pt and ^{192,193}Os nuclei. The Pt and Os nuclei are connected by one-neutron transfer reactions within the same supersymmetric quartet ^{194}Pt $\leftrightarrow$ ^{195}Pt and ^{192}Os $\leftrightarrow$ ^{193}Os, whereas the transitions between the Pt and Os nuclei involve the transfer of a proton pair between different quartets ^{194}Pt $\leftrightarrow$ ^{192}Os and ^{195}Pt $\leftrightarrow$ ^{193}Os.

Generalized F-spin

The correlations between different transfer reactions can be derived in an elegant and explicit way by a generalization of the concept of F-spin which was introduced in the neutron-proton IBM [42] in order to distinguish between proton and neutron bosons.

The eigenstates of the $U(6/12)_\nu \otimes U(6/4)_\pi$ supersymmetry are characterized by the irreducible representations $[N_1, N_2, N_3]$ of $U^{BF_\nu}(6)$ which arise from the coupling of three different $U(6)$ representations, $[N_\nu]$ for the neutron bosons, $[N_\pi]$ for the proton bosons and $[N_\rho]$ for the pseudo-orbital angular momentum of the odd neutron ($N_\rho = 0$ for the even-even and odd-proton nucleus of the quartet, and $N_\rho = 1$ for the odd-neutron and the odd-odd nucleus). In analogy with the three quark flavors in the quark model (u, d and s), also here we have three different types of identical objects (π, ν and ρ), which can be distinguished by F-spin and hypercharge Y. The two kinds of bosons form an F-spin doublet, $F = \frac{1}{2}$, with charge states $F_z = \frac{1}{2}$ for protons (π) and $F_z = -\frac{1}{2}$ for neutrons (ν) [42]. In the framework of the generalized F-spin, we assign in addition a hypercharge quantum number to the bosons $Y = \frac{1}{3}$. The pseudo-orbital part (ρ) has $F = F_z = 0$ and

$Y = -\frac{2}{3}$ [43].

Group theoretically, the generalized F-spin is defined by the reduction

$$\begin{array}{ccccc} U(18) & \supset & U(6) & \otimes & U(3) \\ \downarrow & & \downarrow & & \downarrow \\ [N] & & [N_1,N_2,N_3] & & [N_1,N_2,N_3] \end{array} \tag{87}$$

Here $U(6)$ is to be identified with the $U^{BF_\nu}(6)$ of the group reduction of Eq. (83), which is the result of first coupling the bosons at the level of $U(6)$ followed by coupling the orbital part

$$\left|[N_\nu],[N_\pi];[N_\nu+N_\pi-i,i],[N_\rho];[N_1,N_2,N_3]\right\rangle \;. \tag{88}$$

This sequence of $U(6)$ couplings can be described in a completely equivalent way by the three-dimensional index group $U(3)$ of Eq. (87) which can be reduced to

$$\begin{array}{ccccccccc} U(3) & \supset & SU(3) & \supset & [SU(2) & \supset & SO(2)] & \otimes & U(1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [N_1,N_2,N_3] & & (\lambda,\mu) & & F & & F_z & & Y \end{array} \tag{89}$$

The relation between the two sets of quantum numbers is given by

$$\begin{aligned} (\lambda,\mu) &= (N_1-N_2,N_2-N_3) \;, \\ F &= \frac{1}{2}(N_\pi+N_\nu-2i) \;, \\ F_z &= \frac{1}{2}(N_\pi-N_\nu) \;, \\ Y &= \frac{1}{3}\left(N_\pi+N_\nu-2N_\rho\right) \;. \end{aligned} \tag{90}$$

As a result, matrix elements between states with the same quantum numbers but different $U(6)$ couplings are then related by $SU(3)$ isoscalar factors (or Clebsch-Gordan coefficients for $SU(3)$), and hence correlations between different transfer reactions can be derived in terms of these isoscalar factors by means of the concept of generalized F-spin.

One-neutron transfer

In a study of the ^{194}Pt $\rightarrow$ ^{195}Pt stripping reaction it was found [33] that one-neutron $j = 3/2,\, 5/2$ transfer reactions can be described by the operator

$$P_\nu^{(j)\dagger} \;=\; \frac{\alpha_j}{\sqrt{2}}\left[\left(\tilde{s}_\nu \times a^\dagger_{\nu,j}\right)^{(j)} - \left(\tilde{d}_\nu \times a^\dagger_{\nu,\frac{1}{2}}\right)^{(j)}\right] \;. \tag{91}$$

It is convenient to take ratios of intensities, since they do not depend on the value of the coefficient α_j and hence provide a direct test of the wave functions. For the stripping reaction ^{194}Pt $\to$ ^{195}Pt (ee $\to$ on) the ratio of intensities for the excitation of the $(\tau_1,\tau_2)=(1,0)$, $L=2$ doublet with $J=3/2$, $5/2$ belonging to the first excited band with $[N+1,1]$, $(N+1,1,0)$ relative to that of the ground state band $[N+2]$, $(N+2,0,0)$ is given by [33]

$$R(\mathrm{ee}\to\mathrm{on})=\frac{(N+1)(N+3)(N+6)}{2(N+4)}\ , \tag{92}$$

which gives $R=29.3$ for ^{194}Pt $\to$ ^{195}Pt ($N=5$), to be compared to the experimental value of 19.0 for $j=5/2$, and $R=37.8$ for ^{192}Os $\to$ ^{193}Os ($N=6$). The equivalent ratio for the inverse pick-up reaction is given by

$$R(\mathrm{on}\to\mathrm{ee})=R(\mathrm{ee}\to\mathrm{on})\frac{N_\pi+1}{(N+1)(N_\nu+1)}\ . \tag{93}$$

which gives $R=1.96$ for ^{195}Pt $\to$ ^{194}Pt ($N_\pi=1$ and $N_\nu=4$) and $R=3.24$ for ^{193}Os $\to$ ^{192}Os ($N_\pi=2$ and $N_\nu=4$). This means that the mixed symmetry $L=2$ state is predicted to be excited more strongly than the first excited $L=2$ state.

This correlation between pick-up and stripping reactions has been derived in a general way only using the symmetry relations that exist between the wave functions of the even-even and odd-neutron nuclei of the supersymmetric quartet. The factor in the right-hand side of Eq. (93) is the result of a ratio of two $SU(3)$ isoscalar factors. It is important to emphasize, that Eqs. (92) and (93) are parameter-independent predictions which are a direct consequence of nuclear SUSY and which can be tested experimentally by combining for example $(\vec{d},p)$ stripping and (p,d) pick-up reactions.

Two-proton transfer

The two supersymmetric quartets in the mass $A\sim 190$ region differ by two protons. In principle, the connection between the two quartets can be studied by two-proton transfer

TABLE 4. Ratios of spectroscopic strengths for two-proton transfer reactions between even-even nuclei $R(\mathrm{ee}\to\mathrm{ee})$ to final states with $(\tau_1,\tau_2)=(0,0)$. N is the number of bosons in the odd-odd nucleus of the same quartet as the initial even-even nucleus ($N=5$ for ^{194}Pt $\to$ ^{192}Os).

n	$[N_1,N_2]$	$(\Sigma_1,\Sigma_2,\Sigma_3)$	R_n
1	$[N+3,0]$	$(N+3,0,0)$	1
2	$[N+3,0]$	$(N+1,0,0)$	$\frac{(N+2)(N+5)}{(N+3)^2(N+6)}$
3	$[N+2,1]$	$(N+1,0,0)$	$\frac{(N_\nu+1)(N+1)(N+4)}{(N_\pi+2)(N+3)^2(N+6)}$

TABLE 5. Ratios of spectroscopic strengths for two-proton transfer reactions between odd-neutron nuclei $S(\text{on} \rightarrow \text{on})$ to final states with $(\tau_1,\tau_2)=(0,0)$. N is the number of bosons in the odd-odd nucleus of the same quartet as the initial odd-neutron nucleus ($N=5$ for ^{195}Pt $\rightarrow$ ^{193}Os).

n	$[N+2-i,i]$	$[N_1,N_2]$	$(\Sigma_1,\Sigma_2,\Sigma_3)$	S_n
1	$[N+2,0]$	$[N+3,0]$	$(N+3,0,0)$	1
2	$[N+2,0]$	$[N+3,0]$	$(N+1,0,0)$	R_2
3a	$[N+2,0]$	$[N+2,1]$	$(N+1,0,0)$	$R_3\frac{N_\pi+2}{(N_\nu+1)(N+2)}$
3b	$[N+1,1]$	$[N+2,1]$	$(N+1,0,0)$	$R_3\frac{N_\nu(N+3)}{(N_\nu+1)(N+2)}$

reactions. In the IBM, two-proton transfer operator is, in first order, given by

$$P_\pi^\dagger = \alpha\, s_\pi^\dagger\ , \qquad P_\pi = \alpha\, s_\pi\ . \tag{94}$$

Whereas the operator s_π only excites the ground state of the final nucleus, $s_\pi^\dagger$ can also populate excited states.

In Table 4, we show the results for ratios of spectroscopic strengths between even-even nuclei. The selection rules of the operator $s_\pi^\dagger$ allow the excitation of states with with $(\tau_1,\tau_2)=(0,0)$ and $L=0$ belonging to the ground band $(\Sigma_1,\Sigma_2,\Sigma_3)=(N+3,0,0)$ and excited bands with $(N+1,0,0)$. The corresponding ratios for the odd-neutron nuclei are strongly correlated to those of the even-even nuclei (see Tables 4 and 5)

$$\begin{aligned} S_2(\text{on}\rightarrow\text{on}) &= R_2(\text{ee}\rightarrow\text{ee})\ , \\ S_{3a}(\text{on}\rightarrow\text{on}) &= R_3(\text{ee}\rightarrow\text{ee})\frac{N_\pi+2}{(N_\nu+1)(N+2)}\ , \\ S_{3b}(\text{on}\rightarrow\text{on}) &= R_3(\text{ee}\rightarrow\text{ee})\frac{N_\nu(N+3)}{(N_\nu+1)(N+2)}\ . \end{aligned} \tag{95}$$

As before, the coefficients in the right-hand side correspond to the ratio of two $SU(3)$ Clebsch-Gordan coefficients.

SUMMARY AND CONCLUSIONS

The concept of symmetry has played a very important role in physics, especially in the 20th century with the development of quantum mechanics and quantum field theory. The applications involve among other geometric symmetries, permutation symmetries, space-time symmetries, gauge symmetries and dynamical symmetries. In these lecture notes, I have concentrated mainly on the latter. The basic idea of dynamical symmetries is that of finding order, regularity and simple patterns in complex many-body systems. The examples discussed in these notes include isospin and flavor symmetry and nuclear supersymmetry.

Dynamical symmetries not only provide classification schemes for finite quantal systems and simple benchmarks against which the experimental data can be interpreted in a clear and transparent manner, but also led to important predictions that have been verified later experimentally, such as the Ω^- baryon as the missing member of the baryon decuplet, the nucleus ^{196}Pt as an example of the $SO(6)$ limit of the IBM and the odd-odd nucleus ^{196}Au whose spectroscopic properties had been predicted as a consequence of nuclear supersymmetry almost 15 years before they were measured.

In these lecture notes, I have reviewed the experimental evidence for the existence of supersymmetric quartets of nuclei in the $A \sim 190$ region with $U(6/12)_\nu \otimes U(6/4)_\pi$ supersymmetry, consisting of the ^{194,195}Pt and ^{195,195}Au nuclei, and the ^{192,193}Os and ^{193,194}Ir nuclei, respectively. In addition, nuclear supersymmetry establishes precise links among the spectroscopic properties of different nuclei. This relation has been used to predict the energies of ^{193}Os. Since the wave functions of the members of a supermultiplet are connected by symmetry, there exists a high degree of correlation between different one- and two-nucleon transfer reactions not only between nuclei belonging to the same quartet, but also for nuclei from different multiplets. As an example, the correlations between one-neutron transfer reactions and two-proton transfer reactions were studied.

The interplay between theory and experiment is reflected in the combination of the Platonic ideal of symmetry with the more down-to-earth Aristotelic ability to recognize complex patterns in Nature.

FIGURE 19. Detail of “The School of Athens” (Plato on the left and Aristoteles on the right), by Rafael.

PREHISPANIC SUPERSYMMETRY

FIGURE 20. Prehispanic supersymmetry

Fig. 20 shows an artistic interpretation of supersymmetry in physics. This figure is part of the design of the poster of the *XXXV Latin-American School of Physics. Supersymmetries in Physics and its Applications* (ELAF 2004) by Renato Lemus which is inspired by the concept of supersymmetry as used in nuclear and particle physics and the 'Juego de Pelota', the ritual game of prehispanic cultures of Mexico. The four players on the ballcourt are aztec gods which represent the nuclei of a supersymmetric quartet. Each one of the gods represents a nucleus, on the top left *Tezcatlipoca*: the even-even nucleus ^{194}Pt, top right *Quetzalcóatl*: the odd-even nucleus ^{195}Pt, bottom left *Camaxtle*: the even-odd nucleus ^{195}Au, and finally *Huitzilopochtli*: the odd-odd nucleus ^{196}Au. The association between the gods and the nuclei is made via the number and the color of the balls that each one of the players carry. Each player carries 7 balls. The green and blue balls correspond to the neutron and proton bosons, whereas the yellow and red ones correspond to neutrons and protons, respectively. The one-nucleon transfer operators that induce the supersymmetric transformation between different nuclei, are represented by red coral snakes ('coralillos'). The snakes that create a particle carry a ball in their mouth whose color indicates the type of particle. On the other hand, the snakes that annihilate a particle carry the corresponding ball soaking with blood that seems to split their body. Both types of snakes we see in segmented form, in representation of the quantization of energy. In the world of the ancient Mexico both living and dead creatures form a coherent unity and harmonize in the same plane of importance. This is reflected in the eyes that are included in all components of a graphical representation. For this reason, the balls associated with the creation and annihilation of particles have eyes.

The central figure in the ball court consists of two intertwined snakes, a coral snake and a rattle snake. They represent another aspect of supersymmetry as it is used in particle physics, in which each particle has its supersymmetric counterpart. The reason that this is symbolized by snakes is their property to change skins. Thus, a change of skin of two apparently different snakes suggests the transformation between bosons and fermions. The same two snakes make their appearance on the circular stone rings, the 'score board' of the aztec ball game. In the ball court one finds, at the feet of *Tezcatlipoca* the symbol *Ollin*, movement, which represents the uncertainty principle. Similarly, we see a heart in the upper right and the lower left part. The hearts have two meanings. On the one hand they characterize the ritual aspect of the ancient game 'Juego de Pelota' and, on the other hand, they represent the 'road with a heart', which science could follow. Finally, next to *Huitzilopochtli* there is a skull to remind us of the fleeting nature of our existence.

More information can be found in the proceedings of the ELAF 2004 [44].

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REFERENCES

1. See *e.g. RIA Theory Bluebook: A Road Map* (September 2005), and various contributions to these proceedings.
2. F. Iachello, *Nucl. Phys.* **A751**, 329c (2005).
3. P. Van Isacker, *Rep. Prog. Phys.* **62**, 1661 (1999); Nucl. Phys. **A704**, 232c (2002).
4. A. Frank, J. Barea and R. Bijker, in *The Hispalensis Lectures on Nuclear Physics, Vol. 2*, Eds. J.M. Arias and M. Lozano, Lecture Notes in Physics **652** (2004), pp. 285 [arXiv:nucl-th/0402058].
5. A. Frank, J. Jolie and P. Van Isacker, *Symmetries in Atomic Nuclei: from Isospin to Supersymmetry*, Springer Tracts in Modern Physics **230**, (Springer, 2009).
6. See *e.g.* MacTutor History of Mathematics archive, School of Mathematics and Statistics, University of St. Andrews, Scotland http://www-history.mcs.st-andrews.ac.uk/history
7. W. Pauli, *Z. Phys.* **36**, 336 (1926).
8. M. Hamermesh, *Group Theory and its Applications to Physical Problems*, (Addison Wesley, Reading, 1962 and Dover Publications, New York, 1989).
9. H.J. Lipkin, *Lie Groups for Pedestrians*, (North-Holland, Amsterdam, 1966 and Dover Publications, New York, 2002).
10. R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, (Wiley-Interscience, New York, 1974).
11. B.G. Wybourne, *Classical Groups for Physicists*, (Wiley-Interscience, New York, 1974).
12. J.P. Elliott and P.G. Dawber, *Symmetry in Physics,* (Oxford University Press, Oxford, 1979).
13. H. Georgi, *Lie Algebras in Particle Physics: from Isospin to Unified Theories*, (Addison Wesley, 1982)
14. Fl. Stancu, *Group Theory in Subnuclear Physics,* (Oxford University Press, Oxford, 1996)

15. E.L. Hill, *Rev. Mod. Phys.* **23**, 253 (1951).
16. W. Heisenberg, *Z. Phys.* **77**, 1 (1932).
17. A. Bohr and B.R. Mottelson, *Nuclear Structure. II. Nuclear Deformations* (Benjamin, New York, 1975).
18. M. Gell-Mann, *Phys. Rev.* **125**, 1067 (1962).
19. S. Okubo, *Progr. Theor. Phys.* **27**, 949 (1962).
20. M. Gell-Mann and Y. Ne'eman, *The Eightfold Way*, (Benjamin, New York, 1964).
21. F. Iachello and A. Arima, *The Interacting Boson Model*, (Cambridge University Press, Cambridge, 1987).
22. F. Iachello and P. Van Isacker, *The Interacting Boson-Fermion Model* (Cambridge University Press, Cambridge, 1991).
23. F. Iachello, *Phys. Rev. Lett.* **44**, 772 (1980).
24. A.B. Balantekin, I. Bars and F. Iachello, *Phys. Rev. Lett.* **47**, 19 (1981); A.B. Balantekin, I. Bars and F. Iachello, *Nucl. Phys.* **A370**, 284 (1981).
25. A.B. Balantekin, *et al.*, *Phys. Rev.* C **27**, 1761 (1983).
26. R. Bijker, *Ph.D. Thesis*, University of Groningen (1984).
27. A. Metz, *et al.*, *Phys. Rev. Lett.* **83**, 1542 (1999).
28. A. Metz, *et al.*, *Phys. Rev.* C **61**, 064313 (2000).
29. J. Gröger, *et al.*, *Phys. Rev.* C **62**, 064304 (2000).
30. P. Van Isacker, *et al.*, *Phys. Rev. Lett.* **54**, 653 (1985).
31. J.P. Elliott, *Proc. Roy. Soc.* A **245**, 128 (1958); *ibid.* **245**, 562 (1958).
32. F. Iachello and S. Kuyucak,*Ann. Phys. (N.Y.)* **136**, 19 (1981).
33. R. Bijker and F. Iachello, *Ann. Phys. (N.Y.)* **161**, 360 (1985).
34. H.Z. Sun, A. Frank and P. Van Isacker, *Phys. Rev.* C **27**, 2430 (1983); H.Z. Sun, A. Frank and P. Van Isacker, *Ann. Phys. (N.Y.)* **157**, 183 (1984).
35. M. Balodis *et al.*, *Phys. Rev.* C **77**, 064602 (2008).
36. J.A. Cizewski, *et al.*, *Phys. Rev. Lett.* **40**, 167 (1978); A. Arima and F. Iachello, *Phys. Rev. Lett.* **40**, 385 (1978).
37. J. Barea, R. Bijker and A. Frank, *J. Phys. A: Math. Gen.* **37**, 10251 (2004).
38. H.-F. Wirth, *et al.*, *Phys. Rev.* C **70**, 014610 (2004).
39. J. Barea, R. Bijker and A. Frank, *Phys. Rev. Lett.* **94**, 152501 (2005).
40. J. Jolie and P. Garrett, *Nucl. Phys.* **A596**, 234 (1996).
41. Y. Eisermann, *et al.*, *Czech. J. Phys.* **52**, C627 (2002); H.-F. Wirth, private communication.
42. T. Otsuka, *et al.*, *Phys. Lett.* B **76**, 139 (1978).
43. R. Bijker, J. Barea and A. Frank, *Rev. Mex. Fís.* S **55** (2009), in press [arXiv:0902.4863].
44. R. Lemus, in *Latin-American School of Physics XXXV ELAF. Supersymmetries in Physics and its Applications*, Eds. R. Bijker et al., AIP Conf. Proc. **744** (2005), xi-xvi.