The nuclear impact of atomic physics

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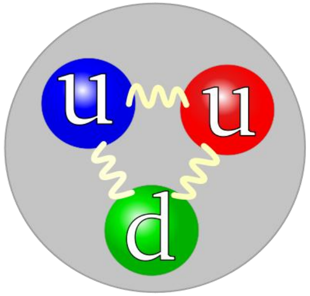


Proton structure

Proton radius: measurements

Proton structure corrections in hydrogen atom

Proton radius: calculations

N. G. Kelkar, F. Garcia Daza, M. Nowakowski, Nucl. Phys B 864, 382 (2012)

SCIENTIFIC AMERICAN™ **The proton shrinks in size**
Tiny change in radius has huge implications.

physorg.com **Particle physics: 'Honey, I shrunk the proton'**

DISCOVER MAGAZINE **The Incredible Shrinking Proton That Could Rattle the Physics World**

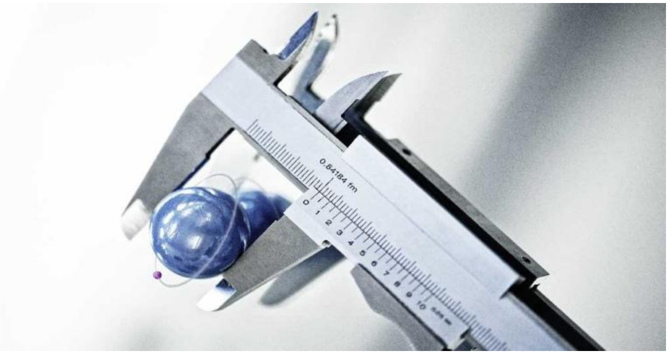
The New York Times **For a Proton, a Little Off the Top (or Side) Could Be Big Trouble**

NewScientist **Incredible shrinking proton raises eyebrows**

Is the new radius redefining physical constants

or

Was it much ado about nothing? Let us find out .....



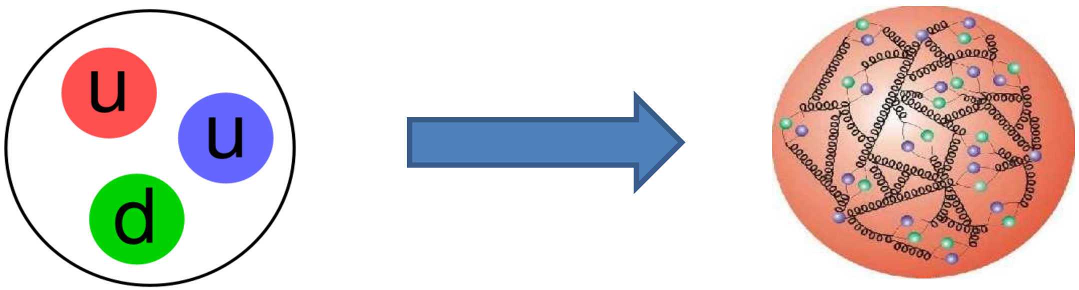
R. Pohl et al, The size of the proton, Nature 466, 213 (2010)

On the basis of a very accurate measurement of the 2S-2P Lamb shift in muonic hydrogen atom and theoretical inputs to calculate the finite size effects due to the proton structure, the charge radius of the proton was deduced to be **0.84184(67) fm**.

This is smaller than the CODATA value of **0.8768(69) fm**

CODATA- Committee on Data for Science and Technology
P. Mohr et al, Rev. Mod. Phys. 80, 633 (2008).

Structure of the proton and electromagnetic form factors



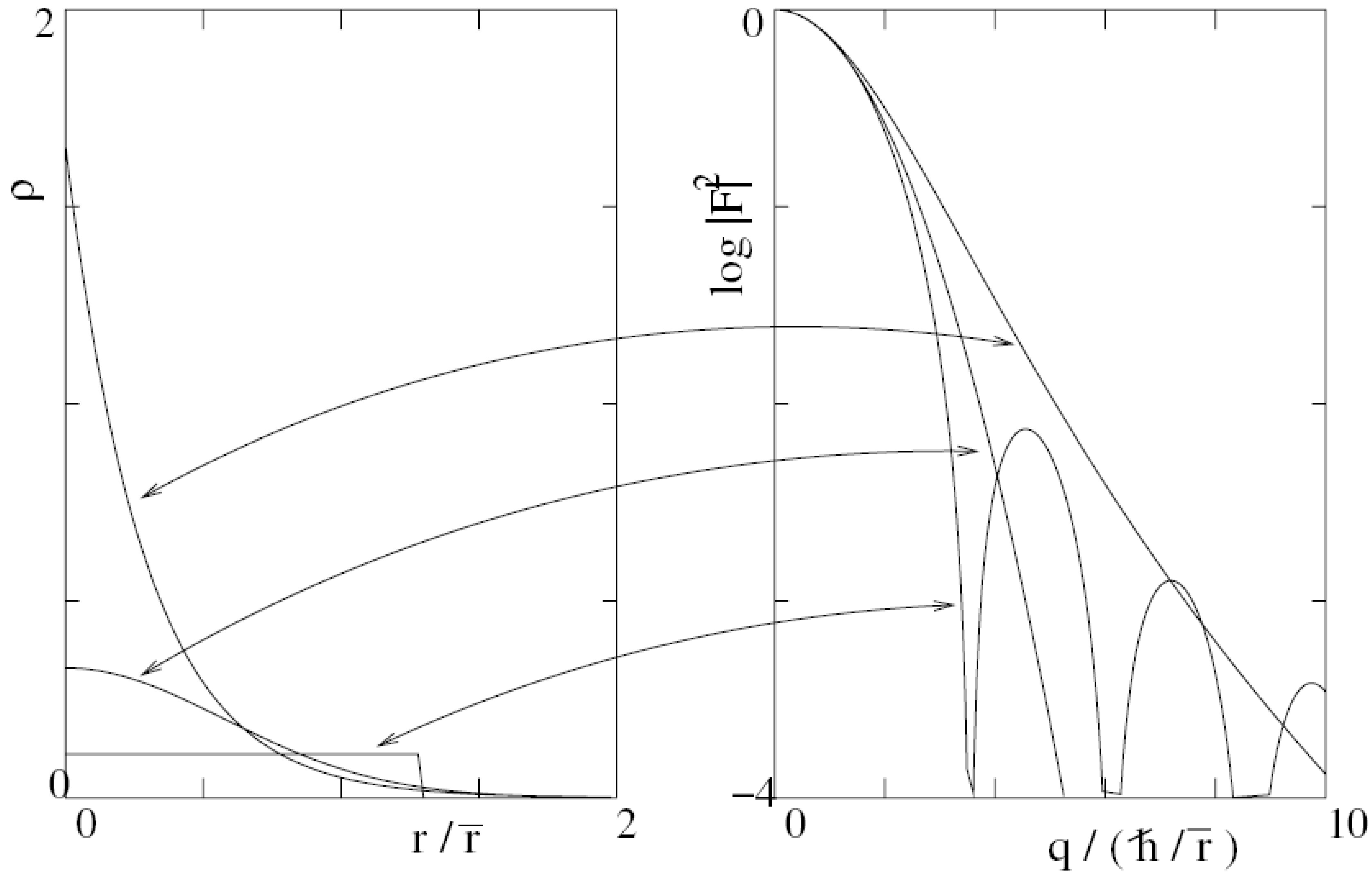
Quarks are charged and hence the charge distribution inside the proton must reflect in electron-proton scattering just as in electron nucleus scattering

For scattering on a charge distribution:

$$\sigma(\theta_e) = \sigma_{Mott} \left| \int_{volume} \rho(\vec{r}) e^{i\vec{q}.\vec{r}} d^3\vec{r} \right|^2 = \sigma_{Mott} |F(\mathrm{q})|^2$$

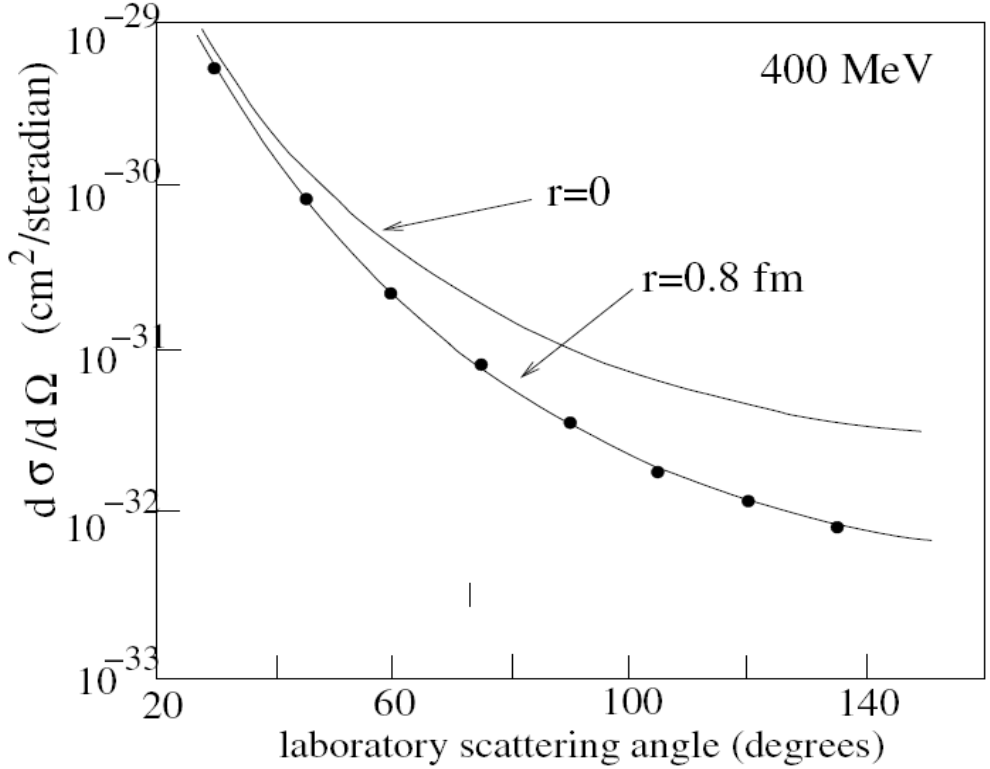
$$[F(\mathrm{q})]^2 = \frac{\sigma(\mathrm{q})}{\sigma_{Mott}(\mathrm{q})}, \quad \text{with} \quad \vec{q} = \vec{p}_{beam} - \vec{p}_e \; \text{ and } \; \mathrm{q} = |\vec{q}|$$

Early nucleon structure investigations – *Hofstader, Rev. Mod Phys. 28, 214 (1956)*



Density distributions: square, Gaussian and exponential

Square: difraction pattern typical of electron-nucleus scattering



400 MeV electrons scattered from protons

Scattering from an exponential charge distribution of rms radius 0.8 fm

$$\rho_p(r) = \rho_0 \mathrm{e}^{-r/a_1}$$

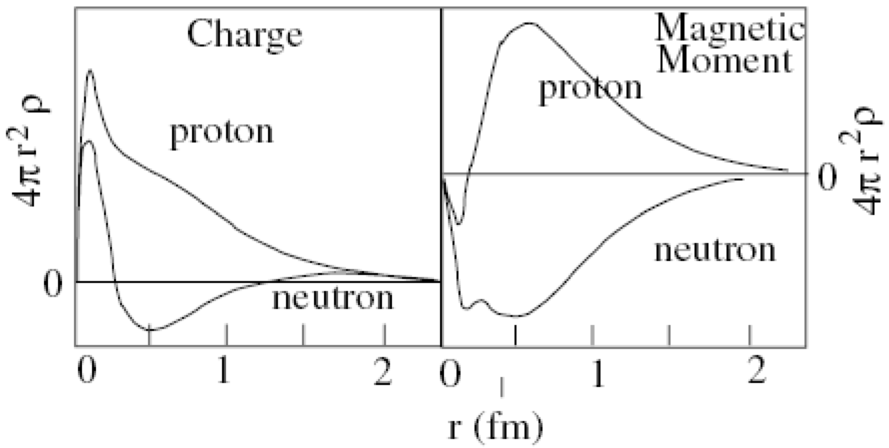
$$\langle r^2 \rangle = 4\pi \int r^4 \rho(r) \mathrm{d}r.$$

which gives a form factor

$$F_p(q) = (1 + q^2 a_1^2/\hbar^2)^{-2}$$

Dirac and Pauli form factors

Charge and magnetization densities



R. M. Littauer et al,, Phys. Rev. Lett. 7, 144 (1961)

For point like spin ½ nucleons

$$\mu_p = \frac{e\hbar}{2m_p} \quad , \quad \text{and} \quad \mu_n = 0 \quad (q_n = 0)$$

Experimental findings give

$$\mu_{\rm p} \;=\; 2.7928444\frac{e\hbar}{2m_p} \qquad \mu_{\rm n} \;=\; -1.91304308\frac{e\hbar}{2m_p}$$

Ernst, Sachs and Wali introduced a set of form factors related to the charge and current distributions in the nucleon

F. J. Ernst, R. G. Sachs and K. C. Wali, Phys. Rev. 119, 1105 (1960)

$$F_{ch} = F_1 - \tfrac{Q^2}{2M} F_2; \qquad F_m = \tfrac{1}{2M} F_1 + F_2.$$

$$\frac{d\sigma}{d\Omega_e} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \frac{E_e}{E_{beam}} \left\{ F_1^2(Q^2) + \tau \left[F_2^2(Q^2) + 2\left(F_1(Q^2) + F_2(Q^2)\right)^2 \tan^2\frac{\theta_e}{2} \right] \right\}$$

$F_1^p(q^2)$ - gives deviation from point charge Dirac particle (Dirac form factor)

$F_2^p(q^2)$ - Deviation from a point anomalous magnetic moment (Pauli form factor)

$$G_E^p(q^2) = F_1^p(q^2) + \frac{q^2}{4m_p^2c^2} F_2^p(q^2),$$
$$G_M^p(q^2) = F_1^p(q^2) + F_2^p(q^2).$$

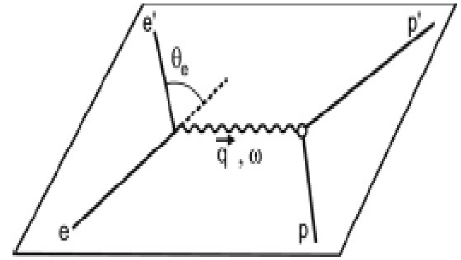
are the so called Sachs form factors, which in the Breit frame are the Fourier transforms of the charge and magnetization densities

$$G_E^p(-\mathbf{q}^2) = \int \rho_C(\mathbf{r})\, e^{-i\mathbf{q}\cdot\mathbf{r}}\, d\mathbf{r}$$
$$G_M^p(-\mathbf{q}^2) = \mu_p \int \rho_M(\mathbf{r})\, e^{-i\mathbf{q}\cdot\mathbf{r}}\, d\mathbf{r}$$

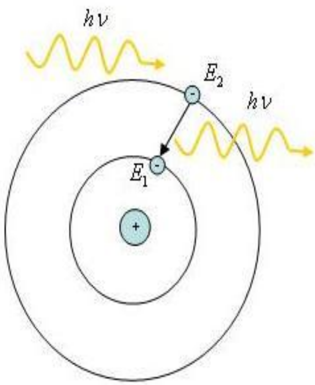
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} \times \left(G_E^2 + \tau\left[1 + 2(1+\tau)\tan^2\frac{\theta_e}{2}\right] G_M^2\right) \Big/ (1+\tau)$$

Proton Radius: How do we measure it?

1. Electron – proton scattering data
   Extract form factors – deduce densities
   and root mean square radius

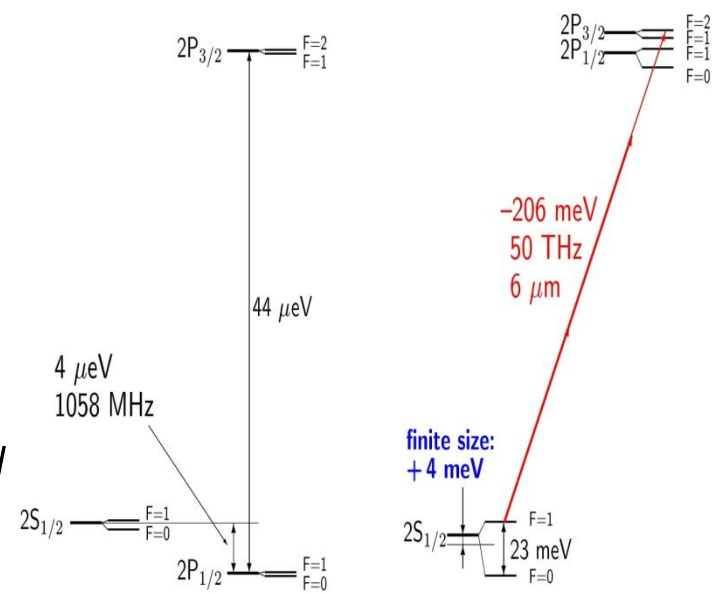


2. Hydrogen spectroscopy
   shielded Coulomb potential -
   shifts in the energy levels due to nuclear structure effects



$$\nu_H(1S_{1/2}\text{-}2S_{1/2}) = 2\,466\,061\,413\,187.074(34)\ \mathrm{kHz}$$
$$[1.4 \times 10^{-14}],$$

3. Muonic hydrogen spectra and Lamb shift
   exotic atom – muon being heavier –
   smaller atom – muon closer to proton
   → finite size effects more prominent



A comparison between n=2, electronic hydrogen (left) and muonic hydrogen levels (right)

Proton radius from e-p scattering experiments

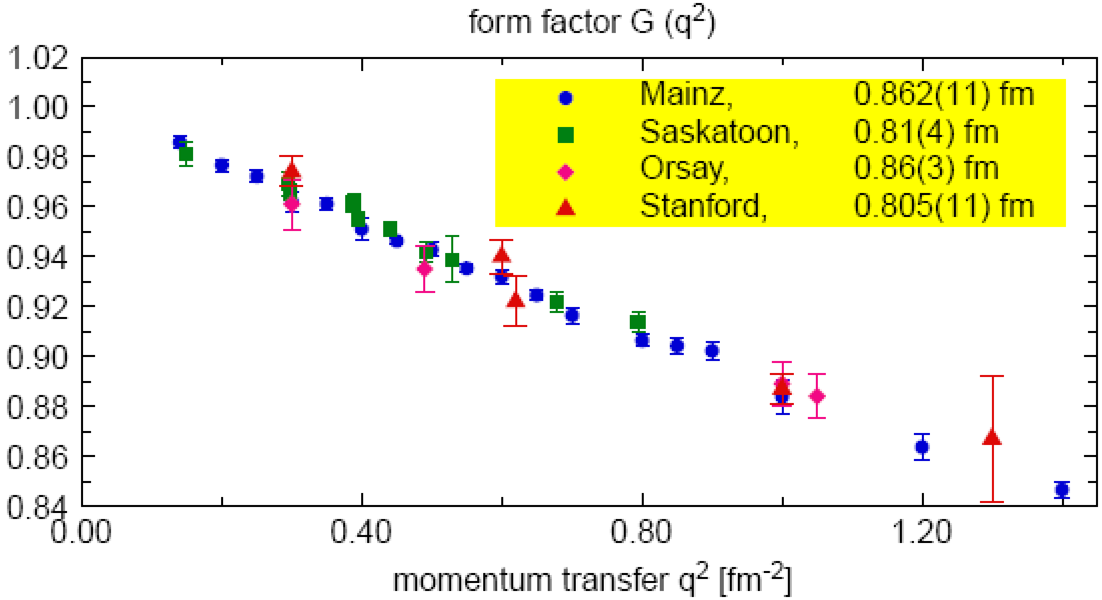
The e-p cross section can be written in terms of the Sachs form factors

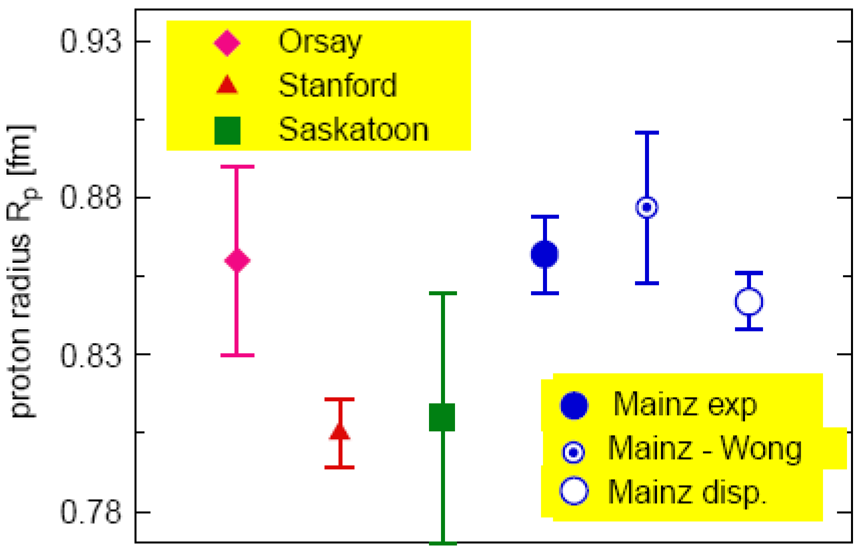
$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{2E_{\text{beam}}\sin(\frac{\theta_e}{2})}\right)^2 \frac{E_e}{E_{\text{beam}}}\left(\frac{\cot^2(\frac{\theta_e}{2})}{1+\tau}[G_E^2 + \tau {G_M}^2] + 2\tau {G_M}^2\right)$$

$$G_E^p(Q^2) = \frac{4\pi}{Q}\int r\,\rho_C(r)\,\sin Qr\,dr \qquad\qquad Q = |\mathbf{q}|$$

$$G_E^p(Q^2) = \frac{4\pi}{Q}\int r\,\rho_C(r)\left[Qr - \frac{Q^3r^3}{6} + \frac{Q^5r^5}{120} + \ldots\right]$$

$$\left.\frac{dG_E^p}{dQ^2}\right|_{Q^2=0} = -\frac{< r_p^2 >}{6}$$





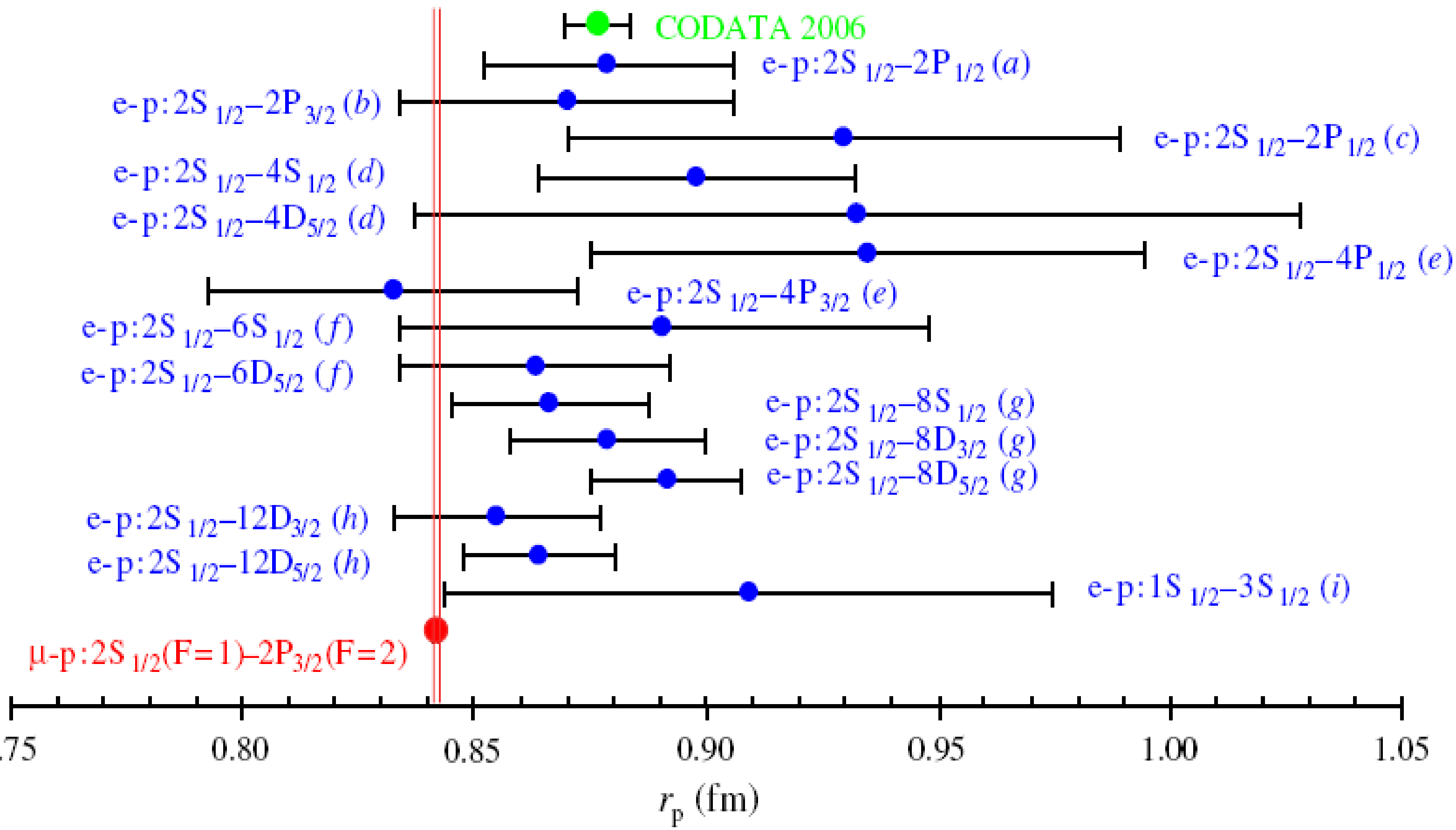
Mainz theoreticians performed a multiparameter fit of all available data using dispersion analysis to obtain a proton radius of **0.847(9)** fm

P. Mergell et al., Nucl. Phys. A 596, 367 (1996)

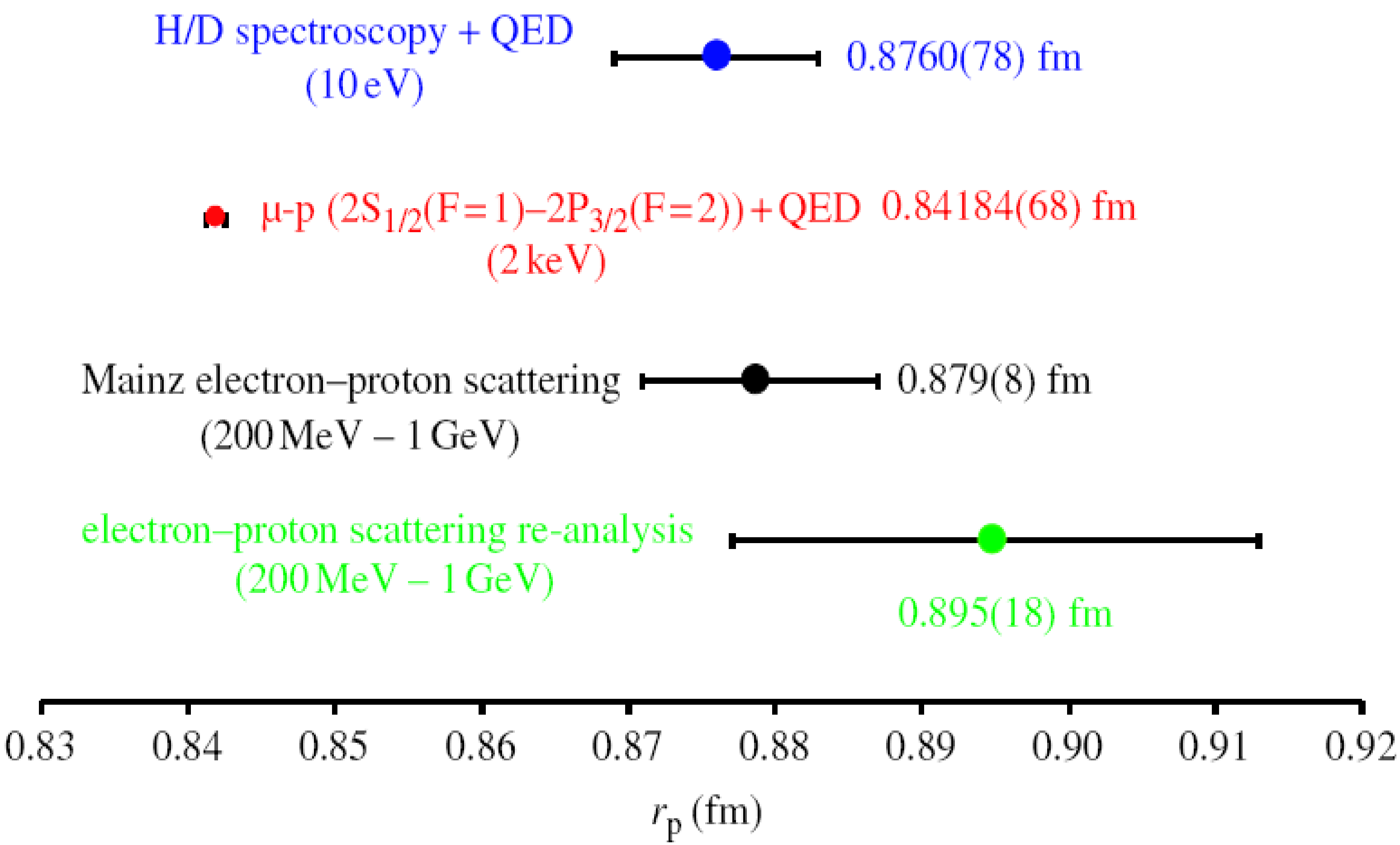
S. G. Karshenboim, Phys. Rep. 422, 1(2005)

Proton radius from hydrogen spectroscopy

Accurate data on the transition frequencies in electronic hydrogen atom. Taken together with QED corrections and finite size effects can be used to determine the radius of the proton.



F. Nez, Phil. Trans. R. Soc. A 369, 4064 (2011)



F. Nez, Phil. Trans. R. Soc. A 369, 4064 (2011)

Rydberg constant , Lamb shift and the proton radius

The Rydberg constant is related to other constants by the relation

$$R_\infty = \alpha^2 \frac{m_\mathrm{e} c}{2h}$$

In the simple Bohr theory, the frequency of transition is proportional to the energy difference between the initial and final states

$$\nu = \frac{E_i - E_f}{h} = \frac{E_0}{h}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

$$\frac{1}{\lambda} = R_\infty \left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

The exact solution of the Dirac equation in the external Coulomb field can be written as follows:

$$E_{nj} = m + m_{\mathrm{r}}[f(n,j) - 1] \ ,$$

$$f(n,j) = \left[1 + \frac{(Z\alpha)^2}{(\sqrt{(j+\frac{1}{2})^2 - (Z\alpha)^2} + n - j - \frac{1}{2})^2}\right]^{-1/2}$$

$$\approx 1 - \frac{(Z\alpha)^2}{2n^2} - \frac{(Z\alpha)^4}{2n^3}\left(\frac{1}{j+(1/2)} - \frac{3}{4n}\right)$$

$$- \frac{(Z\alpha)^6}{8n^3}\left[\frac{1}{(j+(1/2))^3} + \frac{3}{n(j+(1/2))^2} + \frac{5}{2n^3} - \frac{6}{n^2(j+(1/2))}\right] + \cdots \ ,$$

When one includes the recoil corrections and Lamb shift

$$E_{njl}^{\mathrm{tot}} = (m+M) + m_{\mathrm{r}}[f(n,j) - 1] - \frac{m_{\mathrm{r}}^2}{2(m+M)}[f(n,j) - 1]^2 + L_{njl} \equiv E_{nj}^{\mathrm{DR}} + L_{njl}$$

Lamb shift is usually defined as any deviation from the prediction of Dirac equation that arises from radiative, recoil, nuclear structure, relativistic and binding effects but excludes hyperfine contributions

A simpler definition – deviation from the prediction of the Schroedinger equation

The measurement of the Lamb shift can be disentangled from the Rydberg constant by using two different intervals of hydrogen structure. For example,

$$f_{1S-2S} = 2\,466\,061\,413\,187.34(84)\,\mathrm{kHz}, \quad \delta = 3.4 \times 10^{-13},$$

$$f_{2S_{1/2}-8D_{5/2}} = 770\,649\,561\,581.1\,(5.9)\,\mathrm{kHz}, \quad \delta = 7.7 \times 10^{-12}$$

$$E_{1S-2S} = [E^{\mathrm{DR}}_{2S_{1/2}} - E^{\mathrm{DR}}_{1S_{1/2}}] + L_{2S_{1/2}} - L_{1S_{1/2}}$$

$$E_{2S-8D} = [E^{\mathrm{DR}}_{8D_{5/2}} - E^{\mathrm{DR}}_{2S_{1/2}}] + L_{8D_{5/2}} - L_{2S_{1/2}}$$

The first differences on the right hand side are dependent on the Rydberg constant which can be eliminated using the two equations. The left hand side is replaced by accurate measurements and the Lamb shift is determined independent of the Rydberg constant.

Knowing the accurate value of the Lamb shift, the Rydberg constant can be determined

$$R_\infty = 10\,973\,731.568\,527(73) \quad \mathrm{m}^{-1}$$

Measured energy splitting = R_∞ E(n,j) + E(Lamb shift) .... (1)

E(Lamb shift) = E(QED corrections) + E(proton structure)

In R. Pohl et al, The size of the proton, Nature 466, 213 (2010)
For the case of muonic hydrogen atom

$$\Delta E^{\mathrm{exp}}_{2S-2P} = 206.2949(32)\ \mathrm{meV}$$

$$\Delta E^{\mathrm{theo}}_{2S-2P} = 209.9779\,(49) - 5.2262\,r_{\mathrm{p}}^2 + 0.0347\,r_{\mathrm{p}}^3 \quad \mathrm{meV}$$

A comparison of the theoretical and experimental value resulted in the proton radius

0.84184(67) fm

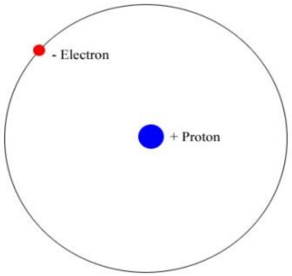
This value is smaller than the CODATA value which is used in (1) to determine the accurate value of the Rydberg constant, R_∞

Plugging the new radius of 0.84184 fm in (1), R_∞ shifts by -110 kHz

Proton structure and the hydrogen atom

Proton in the hydrogen atom is not point-like

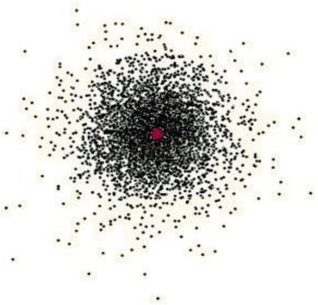
→ 1/R Coulomb potential gets modified



Classically, we can replace the proton by a spherical charge distribution and write

$$V_C(R) = -e^2 \begin{cases} \dfrac{1}{R}, & (R > R_C) \\ \dfrac{1}{2R_C}\left[3 - \left(\dfrac{R}{R_C}\right)^2\right], & (R < R_C) \end{cases},$$

However, a proper calculation has to be quantum mechanical



→ the electron wave function is everywhere!

A simple text book calculation

Itzykson and Zuber – Quantum Field theory

The nucleus has a finite size and its charge distribution is not concentrated at a point. The correction to the energy can be written using the correction to the point Coulomb potential.

$$\Delta E = e \int d\mathbf{r}\, |\Psi(\mathbf{r})|^2 \left[V(r) + \frac{e}{4\pi r} \right]$$

$$\delta V(r) = V(r) - \left(-\frac{e}{r} \right)$$

Approximating the wave function by its value at the origin

$$= e\, |\Psi(0)|^2 \int d\mathbf{r} \left[V(r) + \frac{e}{4\pi r} \right]$$

After some manipulations and using the Poisson equation, $\nabla^2(\delta V(r)) = -\rho_C(r)$

$$\Delta E \simeq \frac{e^2}{6} |\Psi(0)|^2 < r_p >^2$$

Friar´s formalism

J. L. Friar, Annals of Physics 122, 151 (1979)

The 1/r Coulomb potential is modified to

$$\Delta V_c(r) = -Z\alpha \int d^3s\, \rho(s) \left(\frac{1}{|\mathbf{r}-\mathbf{s}|} - \frac{1}{r} \right)$$

$\rho(s)$ - charge density of the nucleus (proton in this case)

Using perturbation theory, the correction to the energy level is

$$\Delta E \simeq \frac{2\pi Z\alpha}{3} \left|\phi_n(0)\right|^2 \left(\langle r^2 \rangle - \frac{Z\alpha\mu}{2} \langle r^3 \rangle_{(2)} + \cdots \right)$$

When one considers the wave function at the origin only, one is left with simple integrals of the type

$$< f(r) > = \int d^3r\, \rho(r)\, f(r)$$

The second term is the third moment of the convoluted proton charge density and is defined as:

$$\langle r^3 \rangle_{(2)} = \int d^3r\, r^3 \rho_{(2)}(r)$$

where the convoluted charge density is given by

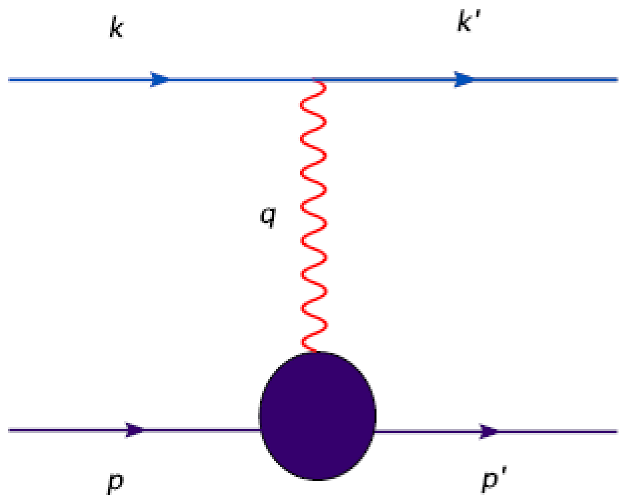
$$\rho_{(2)} = \int d^3z\, \rho_{ch}\big(|\mathbf{z} - \mathbf{r}|\big)\rho_{ch}(z).$$

Inserting the Fourier transform of the electric Sachs form factor

$$\langle r^3 \rangle_{(2)} = \frac{48}{\pi} \int_0^\infty \frac{dQ}{Q^4} \left(G_E^2(Q^2) - 1 + \frac{Q^2}{3} \langle r^2 \rangle \right).$$

Breit potential method for hydrogen atom

F. Garcia Daza, N. G. Kelkar and M. Nowakowski, J. Phys. G39, 035103 (2012)
ibid, AIP Conf. Proc. 1388, 461 (2011)



Standard amplitude for point like protons and electrons

$$M_{fi} = e^2(\bar{u}_1'\gamma^\mu u_1)D_{\mu\nu}(\mathrm{q})(\bar{u}_2'\gamma^\nu u_2)$$

Taking into account the structure of the protons

$$\bar{u}(p')\,\gamma^\mu\,u(p) \longrightarrow \bar{u}(p')\,\Gamma^\mu(p',\,p)\,u(p),$$

$$\bar{u}(p')\,\Gamma^\mu(p',\,p)\,u(p) = \bar{u}(p')\left(\gamma^\mu F_1(q^2) + \frac{i}{2m_p}F_2(q^2)\sigma^{\mu\nu}q_\nu\right)u(p).$$

$$u_i = \sqrt{2m_i}\begin{pmatrix} (1 - \mathbf{p}_i^2/8m_i^2c^2)w_i \\ (\boldsymbol{\sigma}_i.\mathbf{p}_i/2m_ic)w_i \end{pmatrix} \qquad M_{fi} = -2m_e \cdot 2m_p(w_e'^* w_p'^*)\hat{U}(\mathbf{p}_e, \mathbf{p}_p, \mathbf{q})(w_e w_p)$$

$$\hat{U}(\mathbf{p}_X, \mathbf{p}_p, \mathbf{q}) = 4\pi e^2\Bigg[F_1^X F_1^p\Bigg(-\frac{1}{\mathbf{q}^2} + \frac{1}{8m_X^2c^2} + \frac{1}{8m_p^2c^2} + \frac{i\boldsymbol{\sigma}_p.(\mathbf{q}\times\mathbf{p}_p)}{4m_p^2c^2\mathbf{q}^2} - \frac{i\boldsymbol{\sigma}_X.(\mathbf{q}\times\mathbf{p}_X)}{4m_X^2c^2\mathbf{q}^2}$$

$$+\frac{\mathbf{p}_X\cdot\mathbf{p}_p}{m_Xm_pc^2\mathbf{q}^2} - \frac{(\mathbf{p}_X.\mathbf{q})(\mathbf{p}_p.\mathbf{q})}{m_Xm_pc^2\mathbf{q}^4} - \frac{i\boldsymbol{\sigma}_p.(\mathbf{q}\times\mathbf{p}_X)}{2m_Xm_pc^2\mathbf{q}^2} + \frac{i\boldsymbol{\sigma}_X.(\mathbf{q}\times\mathbf{p}_p)}{2m_Xm_pc^2\mathbf{q}^2} + \frac{\boldsymbol{\sigma}_X.\boldsymbol{\sigma}_p}{4m_Xm_pc^2}$$

$$-\frac{(\boldsymbol{\sigma}_X.\mathbf{q})(\boldsymbol{\sigma}_p.\mathbf{q})}{4m_Xm_pc^2\mathbf{q}^2}\Bigg) + F_1^X F_2^p\Bigg(\frac{1}{4m_p^2c^2} + \frac{i\boldsymbol{\sigma}_p.(\mathbf{q}\times\mathbf{p}_p)}{2m_p^2c^2\mathbf{q}^2} - \frac{i\boldsymbol{\sigma}_p.(\mathbf{q}\times\mathbf{p}_X)}{2m_Xm_pc^2\mathbf{q}^2} - \frac{(\boldsymbol{\sigma}_X.\mathbf{q})(\boldsymbol{\sigma}_p.\mathbf{q})}{4m_Xm_pc^2\mathbf{q}^2}$$

$$+\frac{\boldsymbol{\sigma}_X.\boldsymbol{\sigma}_p}{4m_Xm_pc^2}\Bigg) + F_2^X F_1^p\Bigg(\frac{1}{4m_X^2c^2} - \frac{i\boldsymbol{\sigma}_X.(\mathbf{q}\times\mathbf{p}_X)}{2m_X^2c^2\mathbf{q}^2} + \frac{i\boldsymbol{\sigma}_X.(\mathbf{q}\times\mathbf{p}_p)}{2m_Xm_pc^2\mathbf{q}^2} - \frac{(\boldsymbol{\sigma}_X.\mathbf{q})(\boldsymbol{\sigma}_p.\mathbf{q})}{4m_Xm_pc^2\mathbf{q}^2}$$

$$+\frac{\boldsymbol{\sigma}_X.\boldsymbol{\sigma}_p}{4m_Xm_pc^2}\Bigg) + F_2^X F_2^p\Bigg(\frac{\boldsymbol{\sigma}_X.\boldsymbol{\sigma}_p}{4m_Xm_pc^2} - \frac{(\boldsymbol{\sigma}_X.\mathbf{q})(\boldsymbol{\sigma}_p.\mathbf{q})}{4m_Xm_pc^2\mathbf{q}^2}\Bigg)\Bigg]$$

A Fourier transform of this potential can be used to calculate corrections to energies

Similar method has been applied earlier to calculate the finite size corrections in exotic atoms and nuclei – *N. G. Kelkar and M. Nowakowski, PLB 651, 363 (2007), M. Nowakowski, N. G. Kelkar and T. Mart, Phys. Rev. C74, 024323 (2006)*

Potentials in r- space

In order to perform analytic calculations, we use the dipole form factors:

$$G_D(q^2) = 1/(1+q^2/m^2)^2 \approx G_E^p(q^2) \approx G_M^p(q^2)/\mu_p. \qquad (1+\kappa_p) = \mu_p = 2.793$$

Coulomb term:

$$\hat{V}_C^{FF} = -\frac{e^2}{r}\left[1-\left(1+\frac{\kappa_p}{(1-k^2)^2}\right)e^{-mr}-\left(1+\frac{\kappa_p}{(1-k^2)}\right)\frac{m}{2}re^{-mr} + \frac{\kappa_p}{(1-k^2)^2}e^{-mkr}\right],$$

$$k = 2m_p/m$$

Darwin term:

$$\hat{V}_D^{F_{1,2}(q^2)} = \frac{e^2}{8m_X^2c^2}\left[(1+2\kappa_X)G_1+\frac{m_X^2}{m_p^2}G_2\right],$$

$$G_1 = \left(1+\frac{\kappa_p}{1-k^2}\right)\frac{m^3}{2}e^{-mr}+\frac{m^2k^2\kappa_p}{(1-k^2)^2}\frac{e^{-mr}}{r}-\frac{m^2k^2\kappa_p}{(1-k^2)^2}\frac{e^{-mkr}}{r},$$

$$G_2 = \left(1+\kappa_p\left(\frac{1-2k^2}{1-k^2}\right)\right)\frac{m^3}{2}e^{-mr}-\frac{m^2k^2\kappa_p}{(1-k^2)^2}\frac{e^{-mr}}{r}+\frac{m^2k^2\kappa_p}{(1-k^2)^2}\frac{e^{-mkr}}{r}.$$

Darwin term without form factors

$$\hat{V}_D^{\text{no}\,F_{1,2}(q^2)} = \frac{\pi e}{2m_X^2 c}\delta(\mathbf{r})\left[1 + \frac{m_X^2}{m_p^2}\right]$$

Darwin term without form factors at q^2 =0

$$\hat{V}_D^{F_{1,2}(q^2=0)} = \frac{\pi e^2}{2m_X^2 c^2}\delta(\mathbf{r})\left[1 + 2\kappa_X + \frac{m_X^2}{m_p^2}(1+2\kappa_p)\right]$$

Fine structure:

$$\hat{V}_{FS} = \frac{e^2}{4m_X^2 c^2}\left[(1+2\kappa_X) + \frac{2m_X}{m_p}(1+\kappa_X)\right]\left(\frac{1}{r^3} + \frac{G_{FS}}{r^3}\right)\mathbf{L}.\mathbf{S}_X,$$

where,

$$\begin{aligned} G_{FS} \;&=\; -\left(1 + \frac{\kappa_p}{(1-k)^2}\right)e^{-mr}(1+mr) - \left(1 + \frac{\kappa_p}{(1-k)}\right)\frac{m}{2}r^2 e^{-mr} \\ &\quad + \frac{\kappa_p}{(1-k)^2}e^{-mkr}(1+mkr) \end{aligned}$$

Hyperfine structure:

$$\hat{V}_{hfs}(r) = \frac{\alpha\mu_p}{4r^3 m_X m_p c^2}\left[\mu_X\left\{3(\boldsymbol{\sigma}_X\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_p\cdot\hat{\mathbf{r}})f_1(r) - \boldsymbol{\sigma}_X\cdot\boldsymbol{\sigma}_p f_2(r)\right\} + 2\mathbf{L}\cdot\boldsymbol{\sigma}_p f_3(r)\right]$$

$$f_1(r) = 1 - e^{-mr}(1 + mr) - \frac{m^2r^2}{6}e^{-mr}(3 + mr),$$

$$f_2(r) = f_1(r) - (m^3r^3/3)e^{-mr} \text{ and},$$

$$f_3(r) = 1 - e^{-mr}(1 + mr) - \frac{m^2r^2}{2}e^{-mr}.$$

Calculation of corrections to the energies

The energies are evaluated using first order time independent perturbation theory

In general, for any operator $\hat{\mathbf{A}}$

$$\langle \hat{\mathbf{A}} \rangle = \int r^2 \, dr \, d\theta \, d\phi \, \Psi^*_{nlm_l}(r,\theta,\phi) \hat{\mathbf{A}} \Psi_{nlm_l}(r,\theta,\phi),$$

$$\Psi(r,\theta,\phi) = R_{nl}(r) Y_l^{m_l}(\theta,\phi)$$

$$R_{nl}(r) = \left[\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n(n+l)!} \right]^{1/2} e^{-r/na} \left(\frac{2r}{na}\right)^l L^{2l+1}_{n-l-1}(2r/na)$$

$$\begin{aligned} \langle \hat{\mathbf{A}} \rangle \; &= \; \left(\frac{2}{na}\right)^{2l+3} \frac{1}{2n 2^{2(n-l-1)}} \sum_{j=0}^{n-l-1} \binom{2(n-l-j-1)}{n-l-j-1} \frac{(2j)!}{j! \Gamma(2l+j+2)} \\ &\times \int_0^\infty dr \, \mathbf{A} e^{-2r/na} r^{2l+2} L_{2j}^{2(2l+1)}(4r/na). \end{aligned}$$

For example

$$\Delta E_{Coul}(n,l) = \alpha\left(\frac{2}{na_r}\right)^{2l+3}\frac{1}{2n2^{2(n-l-1)}\Gamma(4l+3)}\sum_{j=0}^{n-l-1}\binom{2(n-l-j-1)}{n-l-j-1}\frac{\Gamma(4l+2j+3)}{j!\Gamma(2l+j+2)}$$
$$\times\Bigg[\left(1+\frac{\kappa_p}{(1-k^2)^2}\right)\Gamma(2l+2)\left(\frac{na_r}{2+mna_r}\right)^{2l+2}F\left(-2j,2l+2;4l+3;\frac{4}{2+mna_r}\right)$$
$$-\frac{\kappa_p}{(1-k^2)^2}\Gamma(2l+2)\left(\frac{na_r}{2+mkna_r}\right)^{2l+2}F\left(-2j,2l+2;4l+3;\frac{4}{2+mkna_r}\right)$$
$$+\frac{m}{2}\left(1+\frac{\kappa_p}{1-k^2}\right)\Gamma(2l+3)\left(\frac{na_r}{2+mna_r}\right)^{2l+3}F\left(-2j,2l+3;4l+3;\frac{4}{2+mna_r}\right)\Bigg]$$

In order to get a better insight into the expressions for energies, we replace the hypergeometric functions by the series expansion and truncate the series at large orders of the fine structure constant α

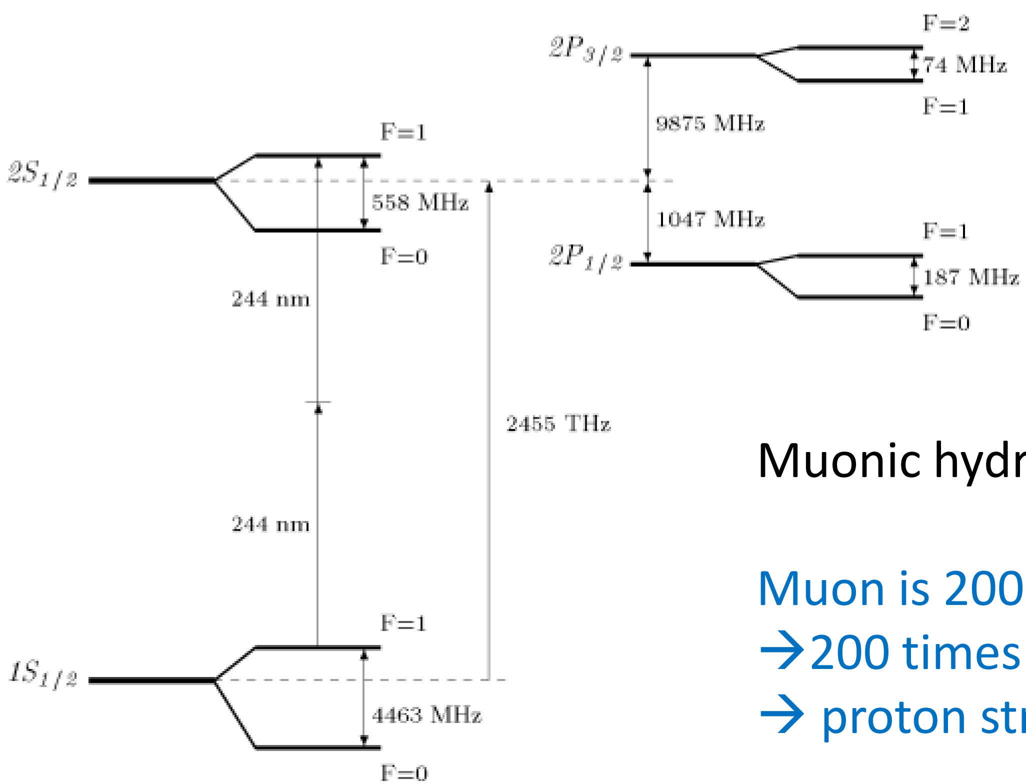
$$\begin{aligned} F(a,b;c;z) &= 1+\frac{ab}{1!c}z+\frac{a(a+1)b(b+1)}{2!c(c+1)}z^2+\dots \\ &= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}. \end{aligned}$$

Proton radius: calculations

Radius calculation of Pohl et al.

R. Pohl et al, The size of the proton, Nature 466, 213 (2010)

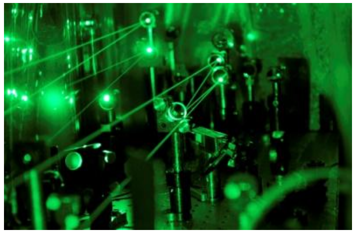
Calculation based on a precise measurement of the Lamb shift in muonic hydrogen



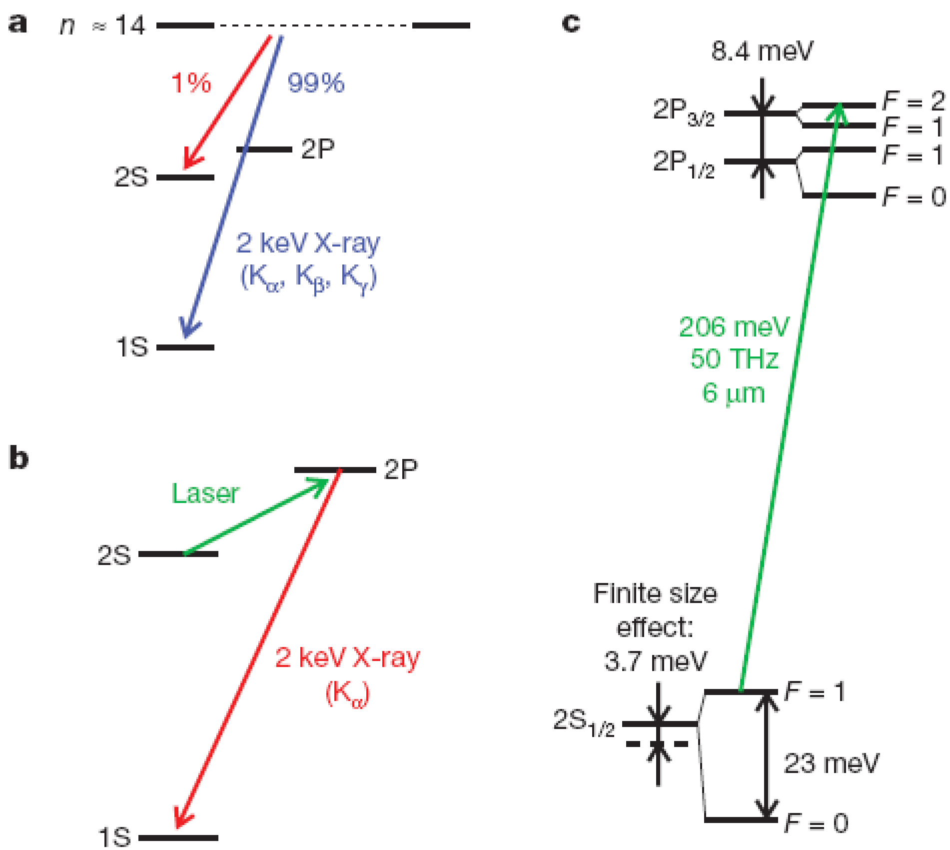
Muonic hydrogen atom energy levels

Muon is 200 times heavier than electron
→200 times smaller Bohr radius
→ proton structure effects enhanced

Experiment was performed at the Paul Scherrer Institute (PSI) Switzerland



They built a new beam line for low energy (~ 5 keV) negative muons
→ order of magnitude more muons than in conventional muon beams

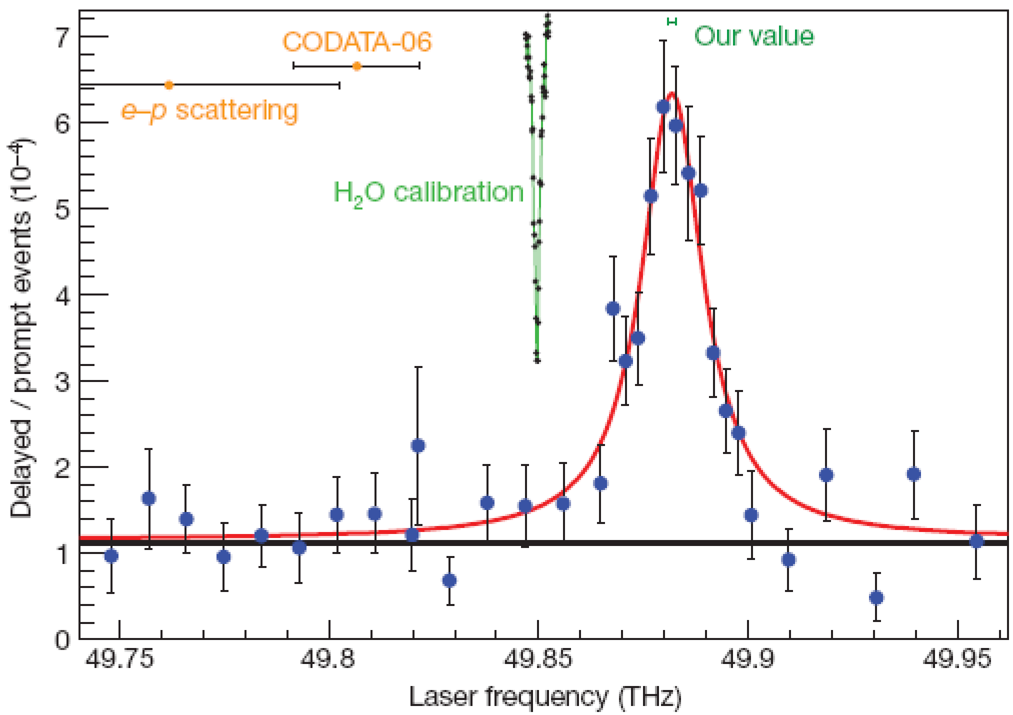


1. Muons are stopped in H_2 gas and highly excited μ-p atoms are formed

2. 99% de-excites quickly to 1S ground state

3. A short laser pulse induces the 2S→ 2P immediately followed by the 2P → 1S de-excitation by emission of 1.9 keV X-rays

4. A resonance curve is obtained by measuring at different laser wavelengths the number of 1.9 keV X-rays that occur in time coincidence with the laser pulse



The centroid position of the $2S_{1/2}^{F=1} - 2P_{3/2}^{F=2}$ transition is 49,881.88(76) GHz, where the uncertainty is the quadratic sum of the statistical (0.70 GHz) and the systematic (0.30 GHz) uncertainties. This frequency corresponds to an energy of $\Delta\tilde{E} = 206.2949(32)$ meV.

$$E_{2P^{F=2}_{3/2}} - E_{2S^{F=1}_{1/2}} = 206.2949(32)\,\mathrm{meV}$$

which has been measured accurately is now compared with the calculated value incorporating the proton structure effects via the proton radius.

$$E_{2P^{F=2}_{3/2}} - E_{2S^{F=1}_{1/2}} = 209.9779\,(49) - 5.2262\,r_{\mathrm{p}}^2 + 0.0347\,r_{\mathrm{p}}^3\ \mathrm{meV},$$

$$\begin{aligned} E_{2S^{f=1}_{1/2}} &= \frac{1}{4}\Delta E^{2S}_{hfs}, \\ E_{2P^{f=2}_{3/2}} &= \Delta E_{LS} + \Delta E^{2P_{3/2}}_{FS} + \frac{3}{8}\Delta E^{2P_{3/2}}_{hfs}, \\ \Delta E_{LS} &= E_{2P_{1/2}} - E_{2S_{1/2}}. \end{aligned}$$

$$\Delta E_{FS} = 8.352082\ \mathrm{meV},$$

$$\Delta E^{2P_{3/2}}_{HFS} = 3.392588\ \mathrm{meV}.$$

$$\Delta E^{2S}_{HFS} = 22.8148\,(78)\ \mathrm{meV}.$$

obtained in A. P. Martynenko, Phys. Rev. A 71, 022506 (2005)
→ Finite size effects included using a Zemach radius 1.022 fm

$$\Delta E_{LS} = 206.0573(45) - 5.2262\,r_{\mathrm{p}}^2 + 0.0347\,r_{\mathrm{p}}^3\ \mathrm{meV},$$

Evaluated using Friar´s formalism for finite size effects

Radius calculation using Breit potential for finite size effects

Comparing an expansion of the Sach´s dipole form factor

$$G_E^p(q^2) = 1 - 2q^2/m^2 + \ldots$$

with the standard expansion of this form factor

$G_E^p(q^2) = 1 - < r_p^2 > q^2/6 + \ldots.$ we can write $< r_p^2 >= 12/m^2$

Replacing for the dipole parameter m by the proton radius as above

$$- \Delta E_{Coul}^{2S_{1/2}} = - (4.30248 r_p^2 - 0.020585 r_p^3)$$

Compare with the terms in Pohl et al., $-5.2262\ r_p^2 + 0.0347\ r_p^3\ \mathrm{meV}$

Different finite size corrections to the 2S energy level

$$\Delta E^{Borie} = \frac{2\alpha Z}{3}\left(\frac{\alpha Z m_r}{n}\right)^3 \left[\langle r^2\rangle - \frac{\alpha Z m_r}{2}\langle r^3\rangle_{(2)} + \cdots\right]$$

E. Borie, Phys. Rev. A 71, 032508 (2005)

Breit potential method gives

$$\Delta E^{2S_{1/2}}_{Coul} = \frac{m_r^3\alpha^4}{24}(A_1 - A_2 + A_3) r_p^2 + \frac{m_r^4\alpha^5}{12\sqrt{12}}\left(-2A_1 + 2\frac{A_2}{k} - 3A_3\right) r_p^3 + \frac{21 m_r^5\alpha^6}{4}\left(A_1 - \frac{A_2}{k^2} + 2A_3\right)\frac{r_p^4}{144} + \cdots,$$

$$A_1 = 1 + [\kappa_p/(1-k^2)^2],\ A_2 = \kappa_p/[k^2(1-k^2)^2],\ A_3 = 1 + [\kappa_p/(1-k^2)]$$

Correction to the Darwin term due to form factors

Once a relativistic calculation using the Dirac equation is performed, one does not need to include the Darwin term … finite size corrections to the Darwin term should however be included!

$$\hat{U}(\mathrm{p}_X,\mathrm{p}_p,\mathrm{q}) = 4\pi e^2\Big[F_1^X F_1^p\Big(-\frac{1}{\mathrm{q}^2}+\frac{1}{8m_X^2c^2}+\frac{1}{8m_p^2c^2}+\frac{i\boldsymbol{\sigma}_p.(\mathrm{q}\times\mathrm{p}_p)}{4m_p^2c^2\mathrm{q}^2}-\frac{i\boldsymbol{\sigma}_X.(\mathrm{q}\times\mathrm{p}_X)}{4m_X^2c^2\mathrm{q}^2}$$

$$+\frac{\mathrm{p}_X.\mathrm{p}_p}{m_Xm_pc^2\mathrm{q}^2}-\frac{(\mathrm{p}_X.\mathrm{q})(\mathrm{p}_p.\mathrm{q})}{m_Xm_pc^2\mathrm{q}^4}-\frac{i\boldsymbol{\sigma}_p.(\mathrm{q}\times\mathrm{p}_X)}{2m_Xm_pc^2\mathrm{q}^2}+\frac{i\boldsymbol{\sigma}_X.(\mathrm{q}\times\mathrm{p}_p)}{2m_Xm_pc^2\mathrm{q}^2}+\frac{\boldsymbol{\sigma}_X.\boldsymbol{\sigma}_p}{4m_Xm_pc^2}$$

$$-\frac{(\boldsymbol{\sigma}_X.\mathrm{q})(\boldsymbol{\sigma}_p.\mathrm{q})}{4m_Xm_pc^2\mathrm{q}^2}\Big)+F_1^XF_2^p\Big(\frac{1}{4m_p^2c^2}+\frac{i\boldsymbol{\sigma}_p.(\mathrm{q}\times\mathrm{p}_p)}{2m_p^2c^2\mathrm{q}^2}-\frac{i\boldsymbol{\sigma}_p.(\mathrm{q}\times\mathrm{p}_X)}{2m_Xm_pc^2\mathrm{q}^2}-\frac{(\boldsymbol{\sigma}_X.\mathrm{q})(\boldsymbol{\sigma}_p.\mathrm{q})}{4m_Xm_pc^2\mathrm{q}^2}$$

$$+\frac{\boldsymbol{\sigma}_X.\boldsymbol{\sigma}_p}{4m_Xm_pc^2}\Big)+F_2^XF_1^p\Big(\frac{1}{4m_X^2c^2}-\frac{i\boldsymbol{\sigma}_X.(\mathrm{q}\times\mathrm{p}_X)}{2m_X^2c^2\mathrm{q}^2}+\frac{i\boldsymbol{\sigma}_X.(\mathrm{q}\times\mathrm{p}_p)}{2m_Xm_pc^2\mathrm{q}^2}-\frac{(\boldsymbol{\sigma}_X.\mathrm{q})(\boldsymbol{\sigma}_p.\mathrm{q})}{4m_Xm_pc^2\mathrm{q}^2}$$

$$+\frac{\boldsymbol{\sigma}_X.\boldsymbol{\sigma}_p}{4m_Xm_pc^2}\Big)+F_2^XF_2^p\Big(\frac{\boldsymbol{\sigma}_X.\boldsymbol{\sigma}_p}{4m_Xm_pc^2}-\frac{(\boldsymbol{\sigma}_X.\mathrm{q})(\boldsymbol{\sigma}_p.\mathrm{q})}{4m_Xm_pc^2\mathrm{q}^2}\Big)\Big]$$

Correction to the Darwin energy terms due to form factors

$$E_D^{F_{1,2}(q^2)} = 14.418512 - 0.0793 r_p + 0.0002613 r_p^2 - 6.6 \times 10^{-6} r_p^3 + \cdots \ \mathrm{meV}$$

$$E_D^{F_{1,2}(q^2=0)} = 14.4185121 \, \mathrm{meV}.$$

$$E_D^{\mathrm{no}\, F_{1,2}(q^2)} = 13.768591 \, \mathrm{meV}$$

$$\Delta E_{Darwin}^{2S_{1/2}} = 0.64992 - 0.0793 r_p + 0.0002613 r_p^2 - 6.6 \times 10^{-7} r_p^3 \ \mathrm{meV},$$

#	Contribution	Ref.	Our selection Value
1	NR One loop electron VP	1,2	
2	Relativistic correction (corrected)	1–3,5	
3	Relativistic one loop VP	5	205.0282
4	NR two-loop electron VP	5,14	1.5081
5	Polarization insertion in two Coulomb lines	1,2,5	0.1509
6	NR three-loop electron VP	11	0.00529
7	Polarisation insertion in two and three Coulomb lines (corrected)	11,12	0.00223
8	Three-loop VP (total, uncorrected)		
9	Wichmann-Kroll	5,15,16	−0.00103
10	Light by light electron loop contribution (Virtual Delbrück scattering)	6	0.00135
11	Radiative photon and electron polarization in the Coulomb line $\alpha^2(Z\alpha)^4$	1,2	−0.00500
12	Electron loop in the radiative photon of order $\alpha^2(Z\alpha)^4$	17–19	−0.00150
13	Mixed electron and muon loops	20	0.00007
14	Hadronic polarization $\alpha(Z\alpha)^4 m_r$	21–23	0.01077
15	Hadronic polarization $\alpha(Z\alpha)^5 m_r$	22,23	0.000047
16	Hadronic polarization in the radiative photon $\alpha^2(Z\alpha)^4 m_r$	22,23	−0.000015
17	Recoil contribution	24	0.05750
18	Recoil finite size	5	0.01300
19	Recoil correction to VP	5	−0.00410
20	Radiative corrections of order $\alpha^n(Z\alpha)^k m_r$	2,7	−0.66770
21	Muon Lamb shift 4th order	5	−0.00169
22	Recoil corrections of order $\alpha(Z\alpha)^5 \frac{m}{M} m_r$	2,5–7	−0.04497
23	Recoil of order α^6	2	0.00030
24	Radiative recoil corrections of order $\alpha(Z\alpha)^n \frac{m}{M} m_r$	1,2,7	−0.00960
25	Nuclear structure correction of order $(Z\alpha)^5$ (Proton polarizability contribution)	2,5,22,25	0.015
26	Polarization operator induced correction to nuclear polarizability $\alpha(Z\alpha)^5 m_r$	23	0.00019
27	Radiative photon induced correction to nuclear polarizability $\alpha(Z\alpha)^5 m_r$	23	−0.00001
	Sum		206.0573

Lamb shift

Radius independent contributions

206.0573 meV

To this are added the finite size corrections depending on the proton radius

$$\Delta E^{Borie} = \frac{2\alpha Z}{3}\left(\frac{\alpha Z m_r}{n}\right)^3 \left[\langle r^2\rangle - \frac{\alpha Z m_r}{2}\langle r^3\rangle_{(2)} + \cdots\right]$$

$$\Delta E_{LS} = E_{2P_{1/2}} - E_{2S_{1/2}}$$

$$\Delta E_{LS} = 206.0573(45) - 5.2262\, r_p^2 + 0.0347\, r_p^3 \text{ meV}$$

is obtained in Pohl et al.

Starting with 206.0573 meV, we also add the finite size corrections to the energy levels in

$$\Delta E_{LS} = E_{2P_{1/2}} - E_{2S_{1/2}}$$

$$\Delta E_{LS} = 205.40738 + 0.0793 r_p - 4.30274 r_p^2 + 0.020585 r_p^3 \text{ meV}$$

$$\Delta E_{LS}^{Nature} = 206.0573(45) - 5.2262 r_p^2 + 0.0347 r_p^3 \text{ meV}$$

The difference in the first two terms is due to the finite size corrections to the Darwin terms which have not been included in the Nature paper.

Coefficients of the r_p^2 and r_p^3 terms

The finite size correction to the Coulomb energy of the present work is evaluated using a potential which is a Fourier transform of

$$U_{Coul}(\mathbf{q}) = 4\pi e^2 F_1^X F_1^p \left(-\frac{1}{\mathbf{q}^2}\right)$$

If we decide to club one of the Darwin terms with the above potential,

$$\begin{aligned} U_{Coul}^{newdef}(\mathbf{q}) &= 4\pi e^2 \left[F_1^X F_1^p \left(-\frac{1}{\mathbf{q}^2}\right) + F_1^X F_2^p \left(\frac{1}{4m_p^2 c^2}\right)\right] \\ &= -4\pi\alpha \left[\frac{G_E^p(\mathbf{q}^2)}{\mathbf{q}^2}\right]. \end{aligned}$$

A Fourier transform of this potential with a dipole form factor gives the potential in r-space as

$$V_C^{newdef} = -\frac{\alpha}{r}\left[1 - e^{-mr}\left(1 + \frac{mr}{2}\right)\right].$$

The energy correction for the 2S state using this new definition is

$$\Delta E_{Coul}^{newdef} = \left(\frac{1}{2a_r}\right)^3 \frac{\alpha}{m^2}\Bigg[\left(\frac{ma_r}{1+ma_r}\right)^2 (1+3F(-2,2;3;2/(1+ma_r))) + \left(\frac{ma_r}{1+ma_r}\right)^3 (1+3F(-2,3;3;2/(1+ma_r)))\Bigg].$$

This after truncating the series expansion of the hypergeometric function leads to

$$\Delta E_{Coul}^{newdef} = \frac{\alpha^4 m_r^3}{12}\left(r_p^2 - \frac{5\alpha m_r}{\sqrt{12}} r_p^3 + \cdots\right).$$

The first term gives exactly the coefficient found in Pohl et al., namely,

$-5.2262 r_p^2$

Taking into account other small corrections

$$\Delta E_{FS}^{2P_{3/2}} = 8.34678 - 4.26 \times 10^{-5} r_p^2 + 1.36 \times 10^{-7} r_p^3 \text{ meV},$$

$$\Delta E_{hfs}^{2P_{3/2}} = 3.3912 - 1.787 \times 10^{-5} r_p^2 + 5.45 \times 10^{-8} r_p^3 \text{ meV},$$

$$(1/4)\Delta E_{hfs}^{2S} = 5.708 - 0.0347 r_p + 0.0001 r_p^2 - 3.27 \times 10^{-7} r_p^3 \text{ meV}$$

$$\boxed{\Delta(= E_{2P_{3/2}}^{f=2} - E_{2S_{1/2}}^{f=1}) = 209.16073 + 0.11388 r_p - 4.3029 r_p^2 + 0.020585 r_p^3 \text{ meV}}$$

$$\boxed{r_p = 0.83112 \text{ fm}}$$

$$\Delta^{Nature} = 209.9779(49) - 5.2262 r_p^2 + 0.0347 r_p^3 \text{ meV}$$

$$r_p = 0.84184(67) \text{ fm}$$

SUMMARY

1. There exist different approaches to evaluate the proton structure effects in the hydrogen atom which lead to different results for the extracted proton radius
   - uncertainty due to approach used
   - in addition to the uncertainty in the determination of proton form factors

2. The Rydberg constant is usually determined from hydrogen spectroscopy. These measurements of transition frequencies are less accurate as compared to the muonic Lamb shift discussed here, but they seem to be highly consistent with each other and with theory.

3. The small radius determined here would shift the Rydberg constant by -110 kHz
   It is related to the electron mass

$$m_{\mathrm{e}} = \frac{2R_\infty h}{c\alpha^2}$$

and also to a new definition of the kilogram $N_{\mathrm{A}} \times h = \frac{c}{2}\,\frac{A_e \alpha^2 M_u}{R_\infty}$

The uncertainty in the experimental determination of N_{A} and h

is three orders of magnitude larger than the shift in the Rydberg constant

4. There is an ongoing experiment to measure the 1S-2S transition frequency in He^+

There is also a plan to measure several transition frequencies between 2S and 2P levels in muonic helium ions by means of laser spectroscopy

A nuclear radius can be extracted from these measurements.
It is expected that a comparison with other data such as e-He+ and μHe^+ scattering would reveal if the discrepancies arise from some missing QED terms and/or bound state QED.